Discriminator Varieties of Double-Heyting Algebras

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Pseudocomplements

Let L be a bounded distributive lattice and let $x \in L$.

- The *pseudocomplement* of x, denoted x^*, is the largest element z such that $z \wedge x = 0$. Equivalently,

$$x \wedge z = 0 \iff z \leq x^*$$

- A *Heyting algebra* is a bounded distributive lattice with an additional operation $\rightarrow$, known as the *relative pseudocomplement*, where $\rightarrow$ satisfies the following equivalence

$$x \wedge z \leq y \iff z \leq x \rightarrow y$$

- In a Heyting algebra, we can define $x^* := x \rightarrow 0$.

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Heyting Algebras

- Recall, the operation $\to$ satisfies the following equivalence

$$x \wedge z \leq y \iff z \leq x \to y$$

- Alternatively, a Heyting algebra is an algebra $\langle H, \vee, \wedge, \to, 0, 1\rangle$ where
  1. $\langle H, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice
  2. $x \to x \approx 1$
  3. $x \wedge (x \to y) \approx x \wedge y$
  4. $x \wedge (y \to z) \approx x \wedge [(x \wedge y) \to (x \wedge z)]$
  5. $z \wedge [(x \wedge y) \to x] \approx z$
- Thus the class of Heyting algebras forms an equational class

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Dual Heyting algebras

- A *dual Heyting Algebra* is simply the dual of a Heyting algebra. The dual of $\rightarrow$ is written $-$ and satisfies the following equivalence

$$x \vee z \geq y \iff z \geq y - x$$

- We also define the *dual pseudocomplement*, x^+, to be the smallest element z such that $x \vee z = 1$. Equivalently,

$$x \vee z = 1 \iff z \geq x^+$$

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Double-Heyting algebras

- An algebra $\langle H, \vee, \wedge, \rightarrow, -, 0, 1 \rangle$ is a *double-Heyting algebra* if
  - $\langle H, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is a *Heyting algebra*
  - $\langle H, \vee, \wedge, -, 0, 1 \rangle$ is a *dual Heyting algebra*

The Discriminator Term

- An algebra A is called a *discriminator algebra* if it has a *discriminator term*, i.e. a term $t(x,y,z)$ where

$$t(x,y,z) = \begin{cases} x & \text{if } x \neq y \\ z & \text{otherwise} \end{cases}$$

- Example: finite fields of order p, we have

$$t(x,y,z) = z + (x-z)(y-x)^{p-1}$$

- A *discriminator variety* is an equational class where there is a term t that is a discriminator term on every subdirectly irreducible member of the class

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The $^{+*}$ operation

Let H be a double-Heyting algebra.

- Recall that the *pseudocomplement* of $x \in H$ is given by $x^* := x \to 0$
- Dually, the *dual pseudocomplement* of $x \in H$ is given by $x^+ := 1 - x$
- We set $x^{0(+*)} = x$, then define $x^{(n+1)(+*)} := (x^{n(+*)})^{+*}$

Lemma

For any x we have

$$x \geq x^{+*} \geq x^{+*+*} \geq \cdots \geq x^{n(+*)} \geq \ldots$$

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Normal filters

Let H be a double-Heyting algebra.

- For a set $F \subseteq H$ we say F is a filter if
  - F is an up-set
  - F is closed under the operation $\wedge$
- If F is also closed under the term operation $^{+*}$ then we say F is a *normal filter on H*
- For any $x \in H$, the normal filter generated by x is given by

$$N(x) = \bigcup_{m \in \omega} \uparrow x^{m(+*)}$$

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Congruences are determined by normal filters

- Let NF(H) denote the lattice of normal filters of H
- For any $F \in \mathrm{NF}(H)$ define the congruence $\theta(F)$ by

$$(x, y) \in \theta(F) \text{ iff } x \wedge f = y \wedge f \text{ for some } f \in F$$

Theorem

The map $\theta : \mathrm{NF}(H) \to \mathrm{Con}(H)$ as given above is an isomorphism.

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Simple implies finite range of $^{+*}$

Lemma

Let H be a double-Heyting algebra. If H is simple, then for every $x \in H$ with $x \neq 1$ there exists some $n_x < \omega$ where $x^{n_x(+*)} = 0$.

Proof.

If H is simple there can only be two normal filters on H. In particular, for any $x \in H$ with $x \neq 1$, we have

$$\begin{aligned} & N(x) = H \\ \iff & 0 \in N(x) \\ \iff & (\exists n_x < \omega)\, 0 \in x^{n_x(+*)} \end{aligned}$$

as $N(x) = \bigcup_{m \in \omega} \uparrow x^{m(+*)}$ □

The class $\mathcal{D}_n$

- The class $\mathcal{D}_n$ is the equational class of double-Heyting algebras satisfying the following equation H

$$x^{(n+1)(+*)} = x^{n(+*)}$$

The class $\mathcal{D}_n$

Theorem

$\mathcal{D}_n$ is a discriminator variety for every $n < \omega$

Proof sketch.

We omit the proof that if $H \in \mathcal{D}_n$ is subdirectly irreducible, then

$$x^{n(+*)} = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

Put $x \leftrightarrow y := (x \to y) \wedge (y \to x)$. The discriminator term is

$$[x \wedge (x \leftrightarrow y)^{n(+*)+}] \vee [z \wedge (x \leftrightarrow y)^{n(+*)}]$$

□

The main result

- An equational class $\mathcal{K}$ is said to be *semisimple* if every subdirectly irreducible algebra in $\mathcal{K}$ is simple.
- It is well-known that every discriminator variety is semisimple. In general, the converse is not true.
- For double-Heyting algebras, it is true

Theorem

Let $\mathcal{V}$ be an equational class of double-Heyting algebras. Then the following are equivalent.

1. *$\mathcal{V}$ is a discriminator variety*
2. *$\mathcal{V}$ is semisimple*
3. *$\mathcal{V} \subseteq \mathcal{D}_n$ for some $n < \omega$*

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