# Discriminator Varieties of Double-Heyting Algebras

## Christopher Taylor

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## Pseudocomplements

Let *L* be a bounded distributive lattice and let *x* ∈ *L*.

The *pseudocomplement* of *x*, denoted *x* ∗ , is the largest element *z* such that  $z \wedge x = 0$ . Equivalently,

## $x \wedge z = 0 \iff z \leq x^*$

A *Heyting algebra* is a bounded distributive lattice with an additional operation →, known as the *relative pseudocomplement*, where  $\rightarrow$  satisfies the following equivalence

$$
x \wedge z \leq y \iff z \leq x \to y
$$

In a Heyting algebra, we can define  $x^* := x \to 0$ .

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# Heyting Algebras

• Recall, the operation  $\rightarrow$  satisfies the following equivalence

### $x \wedge z \leq y \iff z \leq x \rightarrow y$

Alternatively, a Heyting algebra is an algebra  $\langle H, \vee, \wedge, \rightarrow, 0, 1 \rangle$  where

- $\bigoplus \langle H, \vee, \wedge, 0, 1 \rangle$  is a bounded distributive lattice
- 2  $X \rightarrow X \approx 1$

$$
\bullet \quad x \wedge (x \rightarrow y) \approx x \wedge y
$$

 $(4)$  *x*  $\wedge$   $(y \rightarrow z) \approx$  *x*  $\wedge$   $[(x \wedge y) \rightarrow (x \wedge z)]$ 

$$
\mathbf{S} \ \ Z \wedge \left[ (X \wedge y) \rightarrow X \right] \approx Z
$$

Thus the class of Heyting algebras forms an equational class

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# Dual Heyting algebras

A *dual Heyting Algebra* is simply the dual of a Heyting algebra. The dual of  $\rightarrow$  is written – and satisfies the following equivalence

$$
x \vee z \geq y \iff z \geq y - x
$$

We also define the *dual pseudocomplement*, *x* <sup>+</sup>, to be the smallest element *z* such that  $x \vee z = 1$ . Equivalently,

$$
x \vee z = 1 \iff z \geq x^+
$$

In a dual Heyting algebra, we can define  $x^+ := 1 - x$ .

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## Double-Heyting algebras

- **•** An algebra  $\langle H, \vee, \wedge, \rightarrow, -, 0, 1 \rangle$  is a *double-Heyting algebra* if
	- **•**  $\langle H, \vee, \wedge, \rightarrow, 0, 1 \rangle$  is a *Heyting algebra*
	- **•**  $\langle H, \vee, \wedge, -, 0, 1 \rangle$  is a *dual Heyting algebra*

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## The Discriminator Term

An algebra *A* is called a *discriminator algebra* if it has a *discriminator term*, i.e. a term *t*(*x*, *y*, *z*) where

$$
t(x, y, z) = \begin{cases} x & \text{if } x \neq y \\ z & \text{otherwise} \end{cases}
$$

Example: finite fields of order *p*, we have

$$
t(x, y, z) = z + (x - z)(y - x)^{p-1}
$$

A *discriminator variety* is an equational class where there is a term *t* that is a discriminator term on every subdirectly irreducible member of the class

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# The <sup>+</sup><sup>∗</sup> operation

## Let *H* be a double-Heyting algebra.

- Recall that the *pseudocomplement* of  $x \in H$  is given by  $x^* := x \to 0$
- Dually, the *dual pseudocomplement* of  $x \in H$  is given by  $x^+ := 1 - x$
- We set  $x^{0(+*)} = x$ , then define  $x^{(n+1)(++)} := (x^{n(++)})^{+*}$

*For any x we have*

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x \geq x^{+*} \geq x^{+*+*} \geq \cdots \geq x^{n(+*)} \geq \ldots
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### Lemma

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## Normal filters

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## Let *H* be a double-Heyting algebra.

- For a set *F* ⊆ *H* we say *F* is a filter if
	- *F* is an up-set
	- *F* is closed under the operation ∧
- If  $F$  is also closed under the term operation  $^{+*}$  then we say *F* is a *normal filter on H*

For any *x* ∈ *H*, the normal filter generated by *x* is given by

$$
N(x)=\bigcup_{m\in\omega}\uparrow x^{m(+*)}
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## Congruences are determined by normal filters

## Let NF(*H*) denote the lattice of normal filters of *H*

• For any  $F \in \text{NF}(H)$  define the congruence  $\theta(F)$  by

## $(x, y) \in \theta(F)$  iff  $x \wedge f = y \wedge f$  for some  $f \in F$

*The map*  $\theta$  : NF(*H*)  $\rightarrow$  Con(*H*) *as given above is an isomorphism.*

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### Theorem

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# Simple implies finite range of  $^{+*}$

### Lemma

*Let H be a double-Heyting algebra. If H is simple, then for every*  $x \in H$  *with*  $x \neq 1$  *there exists some*  $n_x < \omega$  *where*  $x^{n_x(+)} = 0$ *.* 

### Proof.

If *H* is simple there can only be two normal filters on *H*. In particular, for any  $x \in H$  with  $x \neq 1$ , we have

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$$
N(x) = H
$$
  
\n
$$
\iff 0 \in N(x)
$$
  
\n
$$
\iff (\exists n_x < \omega) \ 0 \in x^{n_x(+*)}
$$

as  $N(x) = \bigcup_{m \in \omega} \uparrow x^{m(+)}$ 

### [The class](#page-27-0) D*n* [The main result](#page-29-0)

# The class D*<sup>n</sup>*

**•** The class  $\mathcal{D}_n$  is the equational class of double-Heyting algebras satisfying the following equation *H*

$$
x^{(n+1)(+*)}=x^{n(+*)}
$$

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## The class D*<sup>n</sup>*

### Theorem

 $D_n$  *is a discriminator variety for every n*  $< \omega$ 

### Proof sketch.

We omit the proof that if  $H \in \mathcal{D}_n$  is subdirectly irreducible, then

$$
x^{n(+*)} = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}
$$

Put  $x \leftrightarrow y := (x \rightarrow y) \land (y \rightarrow x)$ . The discriminator term is

$$
[x\wedge(x\leftrightarrow y)^{n(+*)+}]\vee[z\wedge(x\leftrightarrow y)^{n(+*)}]
$$

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## The main result

- An equational class  $K$  is said to be *semisimple* if every subdirectly irreducible algebra in  $K$  is simple.
- **•** It is well-known that every discriminator variety is semisimple. In general, the converse is not true.
- **•** For double-Heyting algebras, it is true

*Let* V *be an equational class of double-Heyting algebras. Then the following are equivalent.*

- <sup>1</sup> V *is a discriminator variety*
- <sup>2</sup> V *is semisimple*
- 3  $V \subset \mathcal{D}_n$  for some  $n < \omega$

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- <sup>2</sup> V *is semisimple*
	- $\mathcal{V} \subset \mathcal{D}_n$  for some  $n < \omega$

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