Discriminator Varieties of Double-Heyting Algebras

Christopher Taylor

Supervised by Dr. Tomasz Kowalski and Emer. Prof. Brian Davey

Department of Mathematics and Statistics
La Trobe University

Algebra and Substructural Logics 5

Definitions

Let L be a bounded distributive lattice and let $x \in L$.

- The *relative pseudocomplement* operation $x \to y$ satisfies the following equivalence

$$x \wedge z \leq y \iff z \leq x \to y$$

- Dually, the *dual relative pseudocomplement* operation $y - x$ (sometimes written $x \leftarrow y$) satisfies the equivalence

$$x \vee z \geq y \iff z \geq y - x$$

- A *double-Heyting algebra* is a bounded distributive lattice with the additional operations defined above

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The Discriminator Term

- An algebra A is called a *discriminator algebra* if it has a *discriminator term*, i.e. a term $t(x, y, z)$ where

$$t(x,y,z) = \begin{cases} x & \text{if } x \neq y \\ z & \text{otherwise} \end{cases}$$

- A *discriminator variety* is an equational class where there is a term t that is a discriminator term on every subdirectly irreducible member of the class

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The $^{+*}$ operation

Let H be a double-Heyting algebra.

- We can define the *pseudocomplement* of $x \in H$ by $x^* := x \to 0$
- Dually, the *dual pseudocomplement* of $x \in H$ is given by $x^+ := 1 - x$
- We set $x^{0(+*)} = x$, then define $x^{(n+1)(+*)} := (x^{n(+*)})^{+*}$

Lemma

For any x we have

$$x \geq x^{+*} \geq x^{+*+*} \geq \cdots \geq x^{n(+*)} \geq \ldots$$

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Normal filters

- For a set $F \subseteq H$ we say F is a filter if
  - F is an up-set
  - F is closed under the operation $\wedge$
- If F is also closed under the term operation $^{+*}$ then we say F is a *normal filter on H*
- For any $x \in H$, the normal filter generated by x is given by

$$N(x) = \bigcup_{m \in \omega} \uparrow x^{m(+*)}$$

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Congruences are determined by normal filters

Let $\mathrm{NF}(H)$ denote the lattice of normal filters of H

- For any $F \in \mathrm{NF}(H)$ define the congruence $\theta(F)$ by

$$(x, y) \in \theta(F) \text{ iff } x \wedge f = y \wedge f \text{ for some } f \in F$$

Theorem

The map $\theta : \mathrm{NF}(H) \to \mathrm{Con}(H)$ as given above is an isomorphism.

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Theorem

The map $\theta : \mathrm{NF}(H) \to \mathrm{Con}(H)$ *as given above is an isomorphism.*

Simple if and only if $^{+*}$ has finite range

Lemma

Let H be a double-Heyting algebra. Then H is simple if and only if for every $x \in H$ with $x \neq 1$ there exists some $n_x < \omega$ where $x^{n_x(+*)} = 0$.

Proof.

If H is simple there can only be two normal filters on H. In particular, for any $x \in H$ with $x \neq 1$, we have

$$\begin{aligned} N(x) = H &\iff 0 \in N(x) \\ &\iff (\exists n_x < \omega)\ 0 \in {\uparrow} x^{n_x(+*)} \end{aligned}$$

as $N(x) = \bigcup_{m \in \omega} {\uparrow} x^{m(+*)}$ □

The class $\mathcal{D}_n$

- The class $\mathcal{D}_n$ is the equational class of double-Heyting algebras satisfying the following equation H

$$x^{(n+1)(+*)} = x^{n(+*)}$$

The class $\mathcal{D}_n$

Theorem

$\mathcal{D}_n$ is a discriminator variety for every $n < \omega$

Proof sketch.

We omit the proof that if $H \in \mathcal{D}_n$ is subdirectly irreducible, then

$$x^{n(+*)} = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

Put $x \leftrightarrow y := (x \to y) \wedge (y \to x)$. The discriminator term is

$$[x \wedge (x \leftrightarrow y)^{n(+*)+}] \vee [z \wedge (x \leftrightarrow y)^{n(+*)}]$$

□

The main result

- An equational class $\mathcal{K}$ is said to be *semisimple* if every subdirectly irreducible algebra in $\mathcal{K}$ is simple.
- It is well-known that every discriminator variety is semisimple. In general, the converse is not true.
- For double-Heyting algebras, it is true

Theorem

Let $\mathcal{V}$ be an equational class of double-Heyting algebras. Then the following are equivalent.

1. $\mathcal{V}$ *is a discriminator variety*
2. $\mathcal{V}$ *is semisimple*
3. $\mathcal{V} \subseteq \mathcal{D}_n$ *for some* $n < \omega$

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Some required lemmas

Lemma

Let H be a double-Heyting algebra and let $x \in H$. Then for any $k < \omega$ we have

$$x^+ \leq x^{k(+*)+k(+*)}$$

Lemma

Let L be a complete distributive lattice and let $\alpha, \beta \in L$ such that α is compact and α covers β.
Put $\Gamma := \{\gamma \in L \mid \gamma \geq \beta \text{ and } \gamma \not\geq \alpha\}$. Then $\bigvee \Gamma \in \Gamma$.

The main result

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