Algebras of incidence structures: representing regular double p-algebras

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La Trobe University

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Acknowledgements

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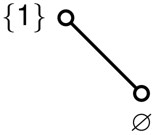

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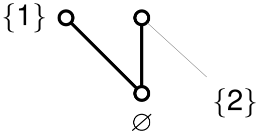

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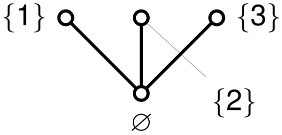

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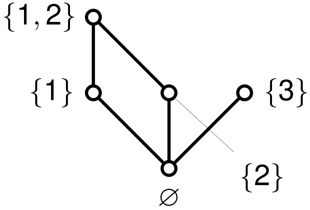

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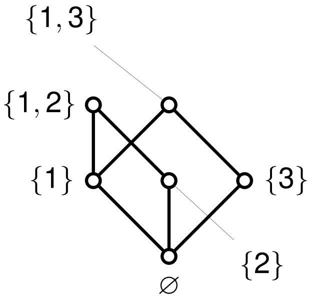

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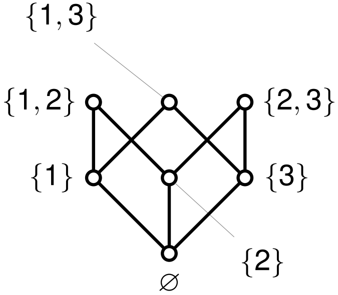

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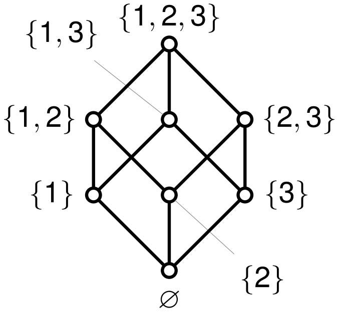

Boolean lattices

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Boolean lattice: a bounded distributive lattice $\mathbf{B} = \langle B; \vee, \wedge, 0, 1 \rangle$ such that every $x \in B$ has a (unique) complement.

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Theorem

Let L be a finite lattice. Then the following are equivalent.

1. *L is a boolean lattice,*
2. *$L \cong \mathcal{P}(B)$ for some finite set B,*
3. *$L \cong \mathbf{2}^n$ for some $n \geq 0$.*

Some other classifications

- Birkhoff's duality for finite distributive lattices
- Stone's duality for boolean algebras
- Priestley's duality for bounded distributive lattices
- Every finite cyclic group is isomorphic to $\mathbb{Z}_n$ for some $n \in \omega$
- Every finite abelian group is isomorphic to $\prod_{i=0}^{n} \mathbb{Z}_{q_i}$ where each q_i is a power of a prime

Graphs

A graph:

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A subgraph:

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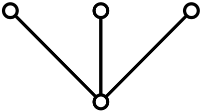

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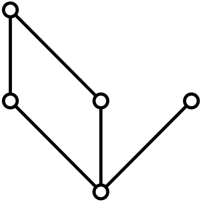

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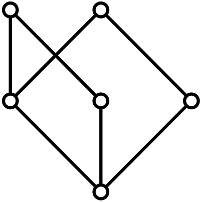

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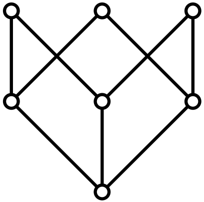

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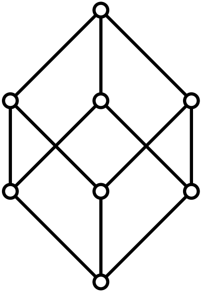

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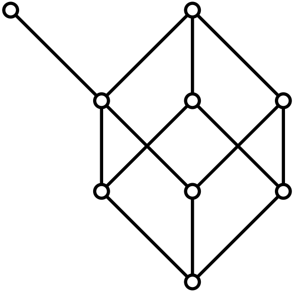

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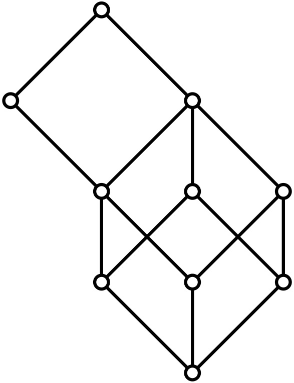

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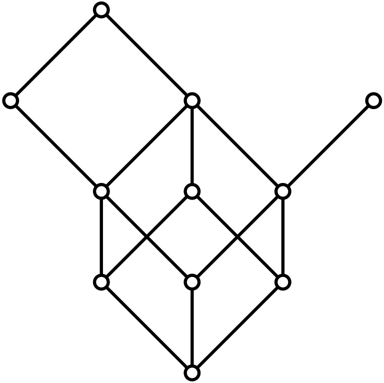

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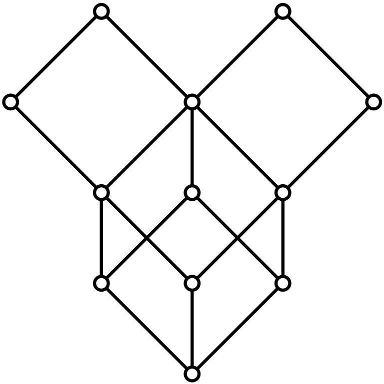

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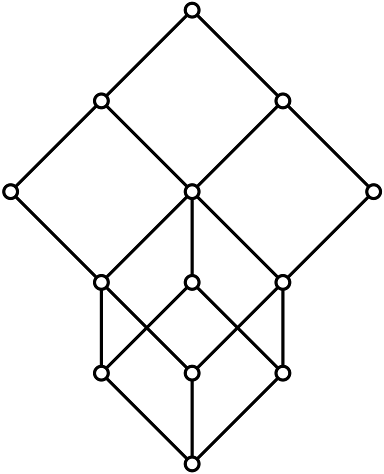


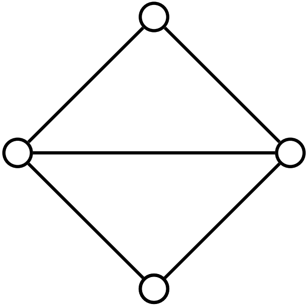
The lattice of subgraphs

- Let $G = \langle V, E \rangle$ be a graph. The set of all subgraphs of G induces a bounded distributive lattice, which we will call $\mathcal{S}(G)$, where

$$\langle V_1, E_1 \rangle \vee \langle V_2, E_2 \rangle = \langle V_1 \cup V_2, E_1 \cup E_2 \rangle$$
$$\langle V_1, E_1 \rangle \wedge \langle V_2, E_2 \rangle = \langle V_1 \cap V_2, E_1 \cap E_2 \rangle.$$

- Note that we permit the empty graph.

Graph complements



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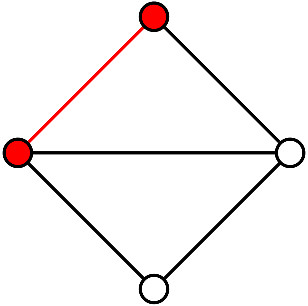

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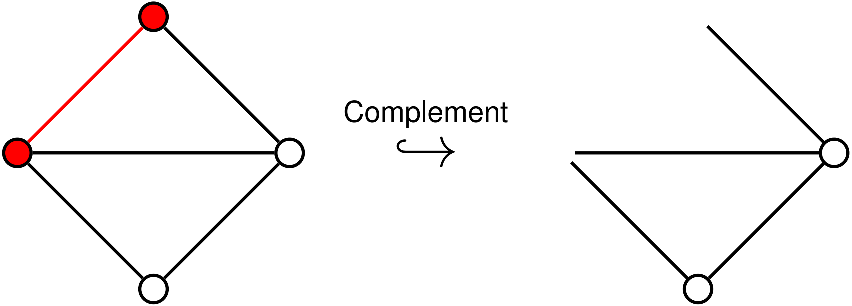

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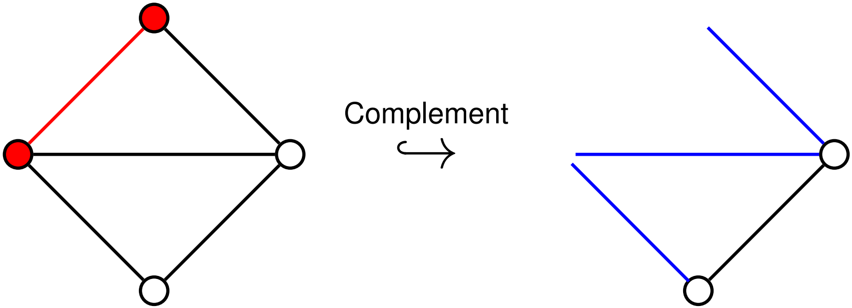

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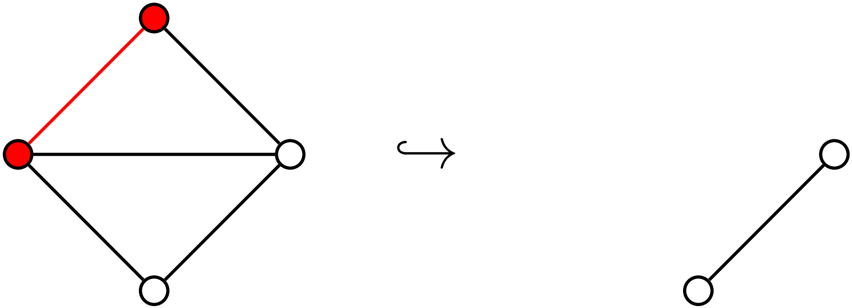

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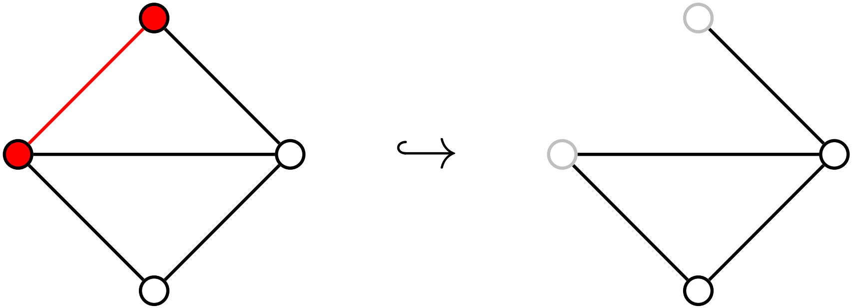

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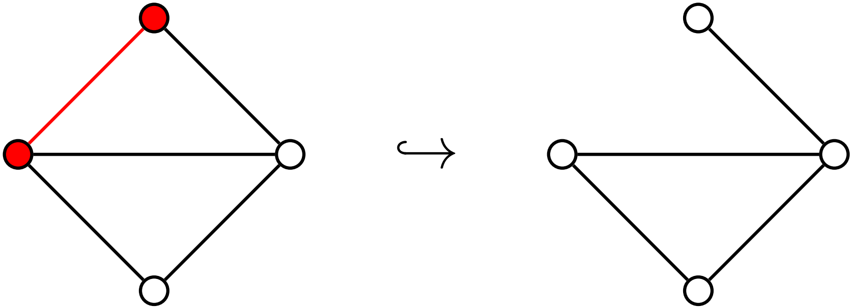

Pseudocomplementation

Let L be a lattice and let $x \in L$. Then x has a *pseudocomplement* if there exists a largest element $x^* \in L$ such that $x \wedge x^* = 0$.

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Definition

An algebra $\mathbf{A} = \langle A; \vee, \wedge, 0, 1, *, + \rangle$ is a *double p-algebra* if $\langle A; \vee, \wedge, 0, 1 \rangle$ is a bounded lattice, and * and $^+$ are the pseudocomplement and dual pseudocomplement respectively.

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Take the set complement of the subgraph and abandon the extra edges.

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- Formally, for a graph $G = \langle V, E \rangle$ and a subgraph $H = \langle V', E' \rangle$:

$$H^* = \langle V \backslash V', \{e \in E \backslash E' \mid (\forall x \in e)\ x \in V \backslash V'\} \rangle$$
$$H^+ = \langle V \backslash V' \cup \{v \in V \mid (\exists e \in E \backslash E')\ v \in e\}, E \backslash E' \rangle.$$

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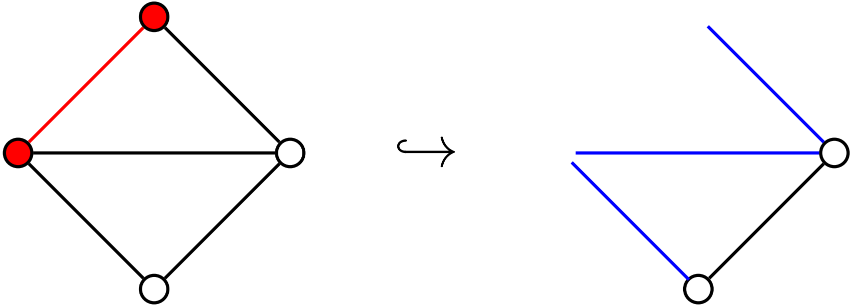

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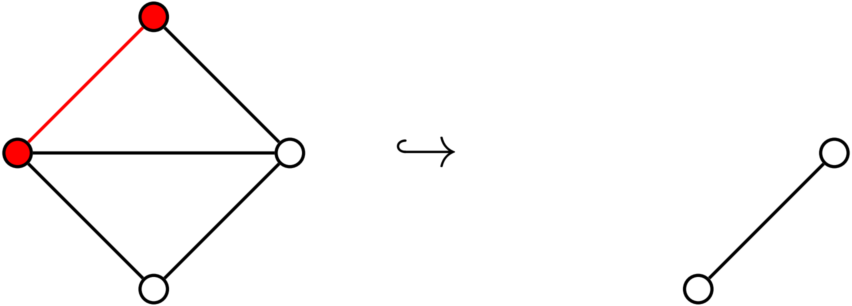
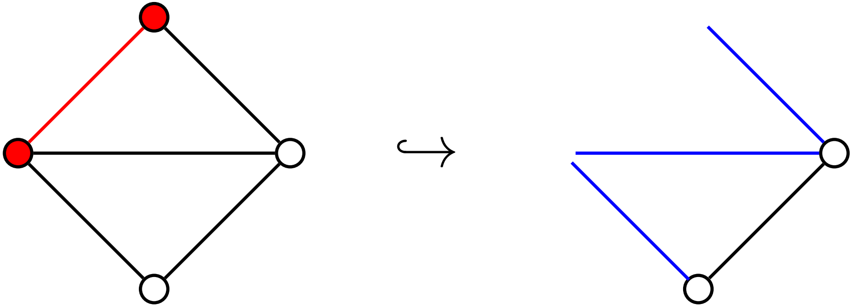


Pseudocomplements are not bijective

Boolean lattices: no two elements share a complement
Double p-algebras: not true!



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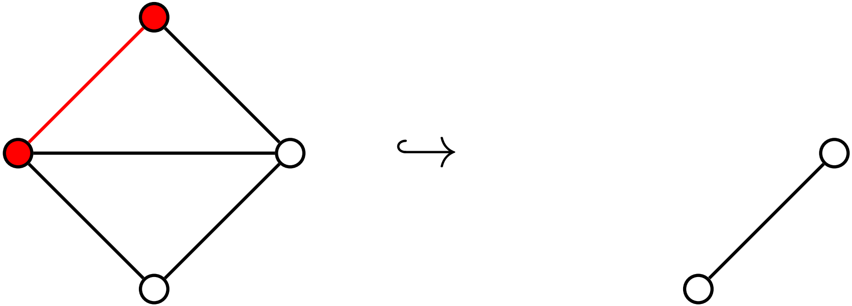

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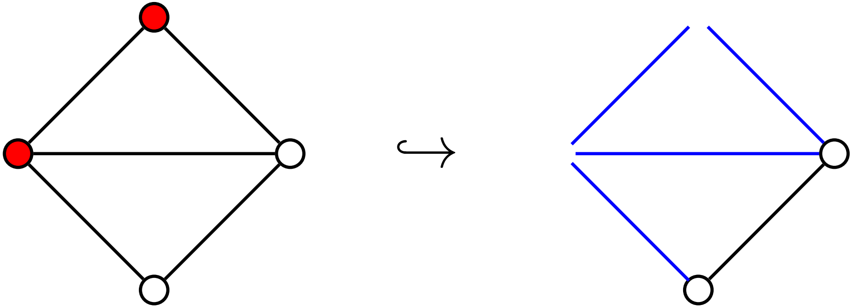

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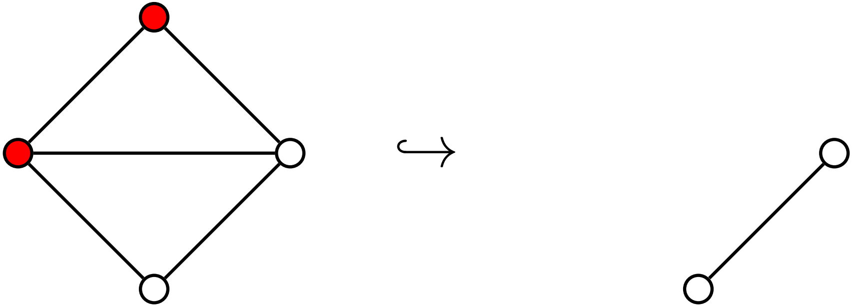

Regular double p-algebras

- Let **A** be an algebra. We say that **A** is *congruence regular* if, for all $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$, we have

$$((\exists x \in A)\ x/\alpha = x/\beta) \implies \alpha = \beta.$$

- Example: groups

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- Example: groups

Theorem (Varlet, 1972)

Let **A** *be a double p-algebra. Then the following are equivalent.*

1. **A** *is congruence regular.*
2. $(\forall a, b \in A)$ *if* $a^* = b^*$ *and* $a^+ = b^+$ *then* $a = b$.
3. $(\forall a, b \in A)\ a \wedge a^+ \leq b \vee b^*$.

A well-behaved structure

Theorem

Let $G = \langle V, E \rangle$ be a graph. Then $\mathcal{S}(G)$ is (the underlying lattice of) a regular double p-algebra.

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Proof.

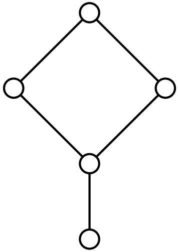
Let $A = \langle A_V, A_E \rangle$ and $B = \langle B_V, B_E \rangle$ be subgraphs of G. Recall that for a subgraph $H = \langle V', E' \rangle$,

$$H^* = \langle V \backslash V', \{e \in E \backslash E' \mid (\forall x \in e)\ x \in V \backslash V'\} \rangle \qquad (1)$$

$$H^+ = \langle V \backslash V' \cup \{v \in V \mid (\exists e \in E \backslash E')\ v \in e\}, E \backslash E' \rangle. \qquad (2)$$

Assume $A^* = B^*$ and $A^+ = B^+$. Then from (1) we have $V \backslash A_V = V \backslash B_V$ and from (2) we have $E \backslash A_E = E \backslash B_E$. Hence, $A = B$. □

Are graphs enough?



$\rightarrow$

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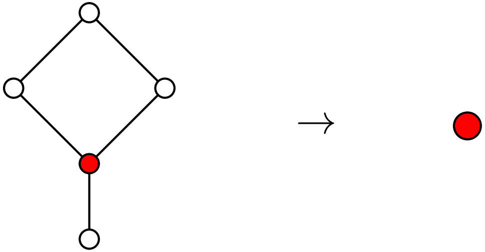

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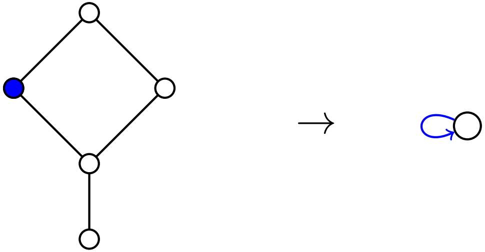

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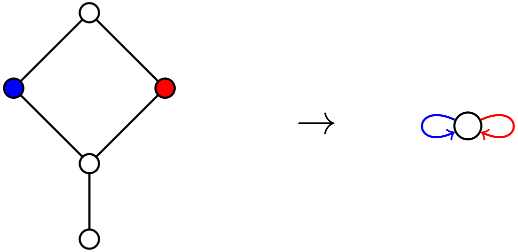

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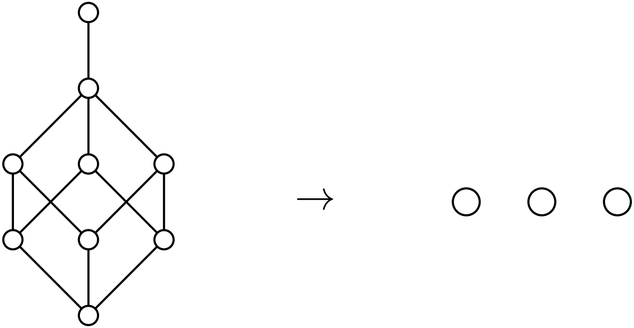

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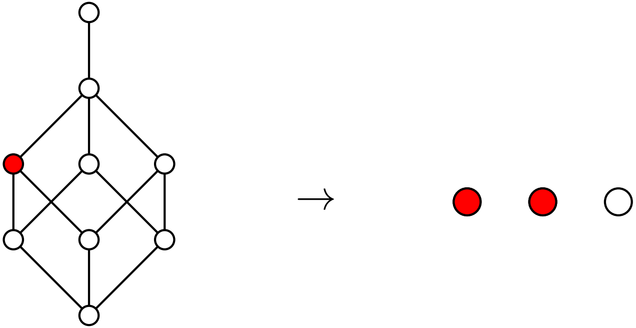

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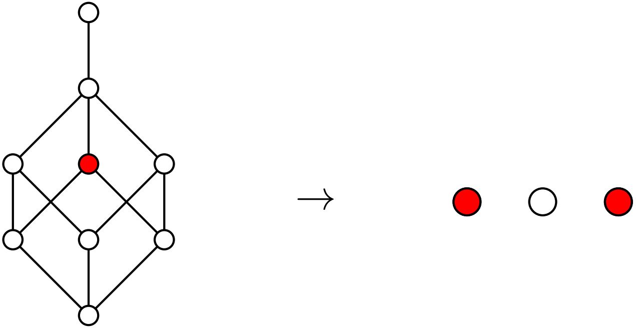

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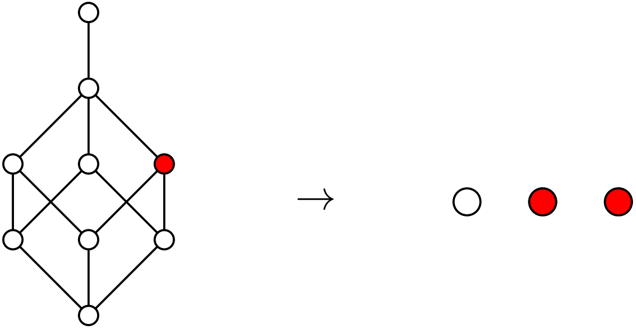

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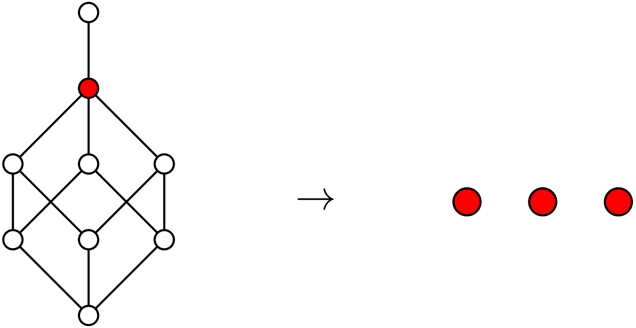

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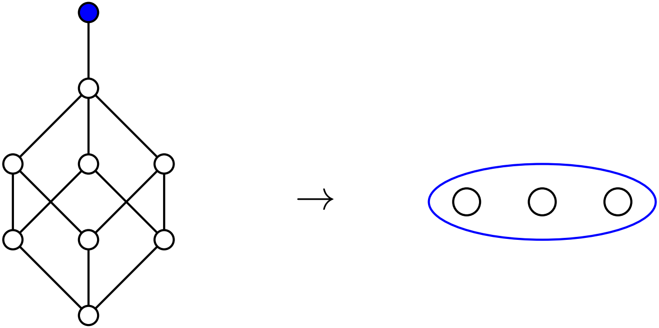

Incidence structures

Definition

An *incidence structure* is a triple $\langle P, L, I\rangle$ where P is a set of points, L is a set of lines and $I \subseteq P \times L$ is an incidence relation describing which points are incident to which lines.

Example

Let $P = \{1, 2, 3\}$, $L = \{x, y, z, a, b\}$, and let

$$\begin{aligned} I = {} & \{1, 2, 3\} \times \{x, y\} \\ & \cup \{1, 2\} \times \{z\} \\ & \cup \{(1, a), (1, b)\} \end{aligned}$$

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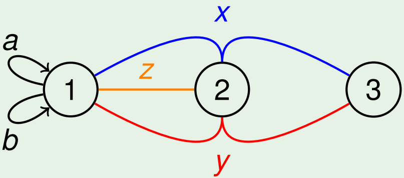

Point-preserving substructures

Definition

Let $G = \langle P, L, I \rangle$ be an incidence structure. A *point-preserving substructure* of G is a pair $\langle P', L' \rangle$ such that

1. $P' \subseteq P$ and $L' \subseteq L$,
2. for all $\ell \in L'$, if $(p, \ell) \in I$ then $p \in P'$.

The incidence relation is defined implicitly from G.

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Let $\mathcal{S}(G)$ denote the set of all point-preserving substructures of a structure G. This induces a double p-algebra in a similar way to graphs, where

$$\langle P', L' \rangle^* = \langle P \backslash P', \{\ell \in L \backslash L' \mid (\forall p \in P)\ (p, \ell) \in I \implies p \in P \backslash P'\} \rangle$$
$$\langle P', L' \rangle^+ = \langle P \backslash P' \cup \{p \in P \mid (\exists \ell \in L \backslash L')\ (p, \ell) \in I\}, L \backslash L' \rangle.$$

The main result (finite version)

Theorem

Let L be a finite lattice. Then the following are equivalent.

1. *L is a boolean lattice,*
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Theorem (Taylor, 2015)

Let L be a finite lattice. Then the following are equivalent.

1. *L is (the underlying lattice of) a regular double p-algebra,*
2. *$L \cong \mathcal{S}(G)$ for some incidence structure G,*
3. *$L \cong \mathbf{2}^n \times \mathcal{S}(G)$ for some $n \geq 0$ and some incidence structure G.*

An infinite counterexample

- The *finite-cofinite algebra* of $\mathbb{N}$ is a boolean algebra.
  - $\mathrm{FC}(\mathbb{N}) := \{S \subseteq \mathbb{N} \mid S \text{ is finite or } \mathbb{N} \setminus S \text{ is finite}\}$.

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- Every powerset lattice has cardinality 2^X for some set X.
- Thus $\mathsf{FC}(\mathbb{N})$ is not a powerset algebra.

The characterisation of powerset algebras

Theorem

Let B be a boolean lattice. Then the following are equivalent.

1. $B \cong \mathcal{P}(X)$ *for some set* X.
2. B *is complete and atomic.*
3. B *is complete and completely distributive.*

The main result

Theorem (Taylor, 2015)

Let **A** *be a regular double p-algebra. Then the following are equivalent.*

1. $\mathbf{A} \cong \mathcal{P}(B) \times \mathcal{S}(G)$ *for some set* B *and some incidence structure* G*.*
2. $\mathbf{A} \cong \mathcal{S}(G)$ *for some incidence structure* G*.*
3. **A** *is complete, completely distributive and doubly atomic.*

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2. $\mathbf{A} \cong \mathcal{S}(G)$ *for some incidence structure* G.
3. **A** *is complete, completely distributive and doubly atomic.*

Theorem (Taylor, 2015)

Let **A** *be a regular double p-algebra. Then there is an incidence structure* G *such that* **A** *is isomorphic to a subalgebra of* $\mathcal{S}(G)$.