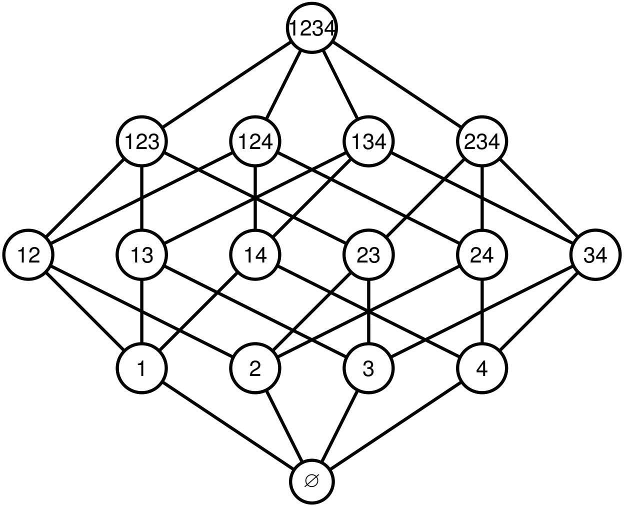
Expansions of dually pseudocomplemented Heyting algebras

Christopher Taylor

Supervised by Tomasz Kowalski and Brian Davey

AustMS 2016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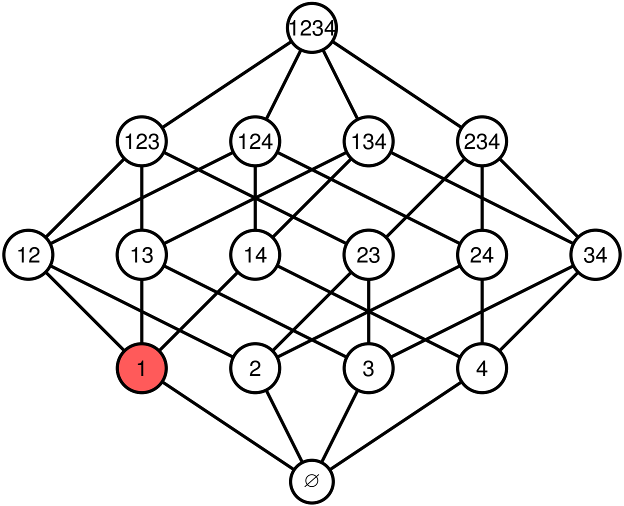
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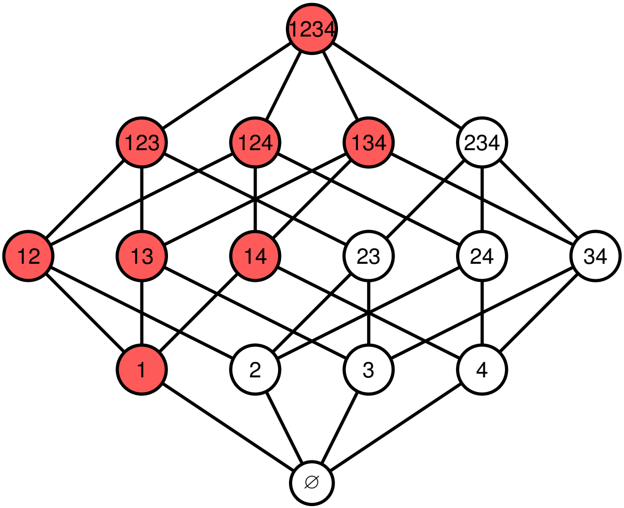
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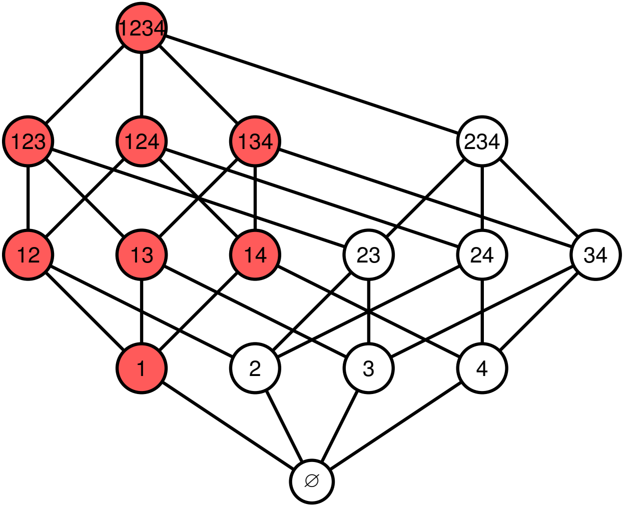
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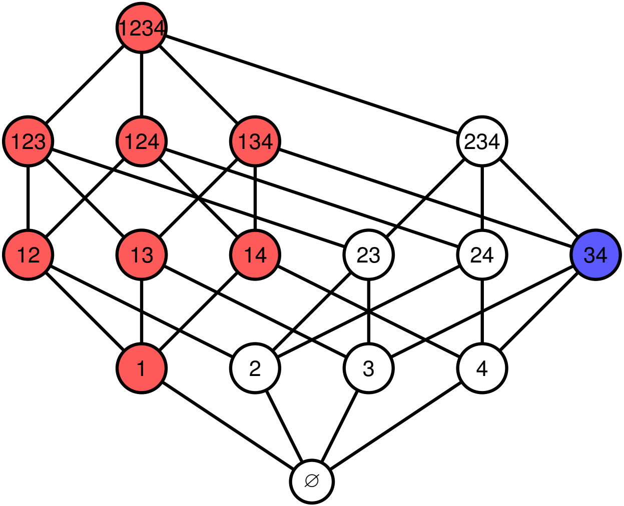
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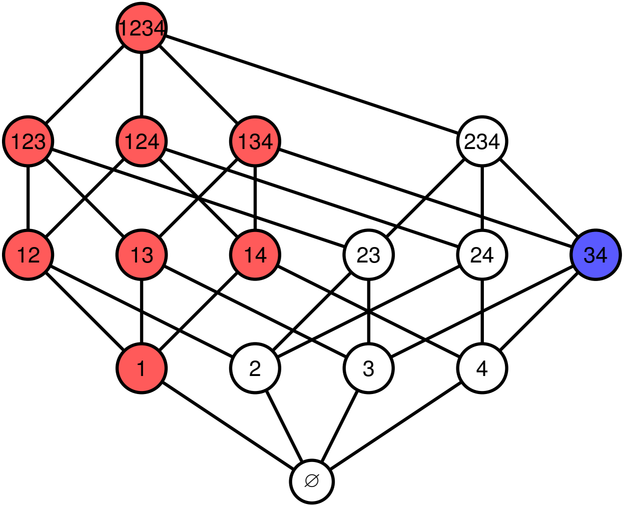
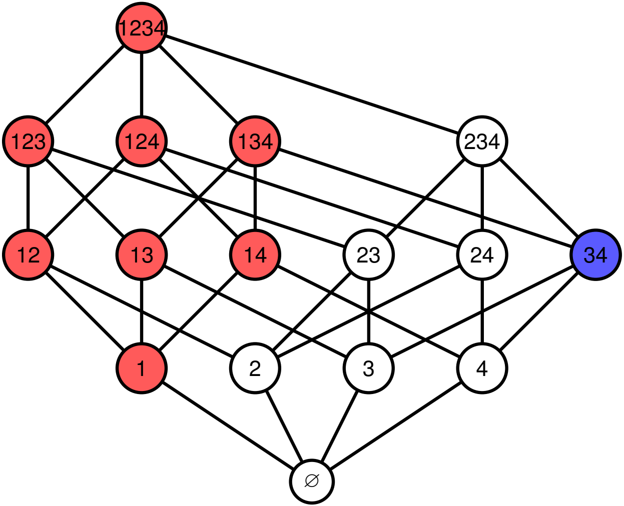


Relative complement of 12 with respect to 1 is equal to $1 \vee 34$.



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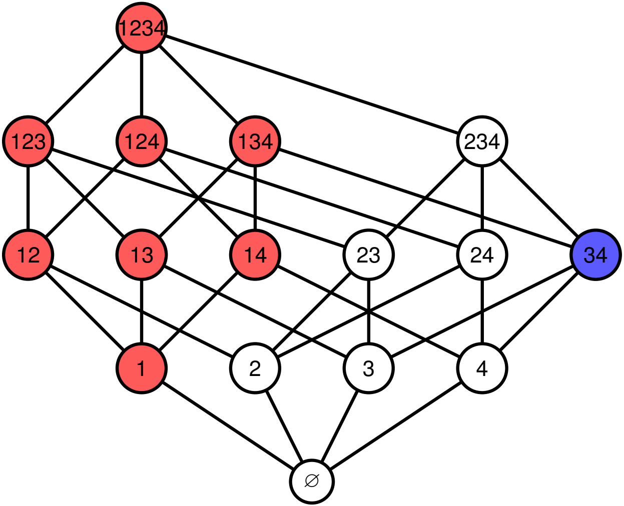

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...also known as $x \to y$

Logical interpretation

- Implication:

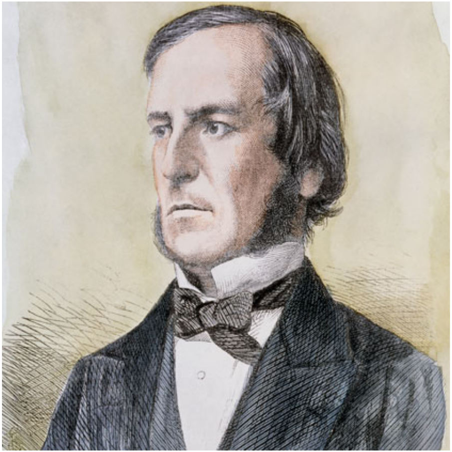
$$p \to q := \neg p \vee q$$

- Law of the excluded middle:

$$p \vee \neg p = 1$$

- Law of non-contradiction:

$$p \wedge \neg p = 0$$



George Boole

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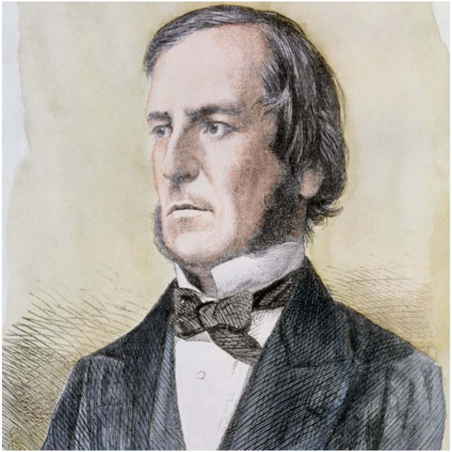

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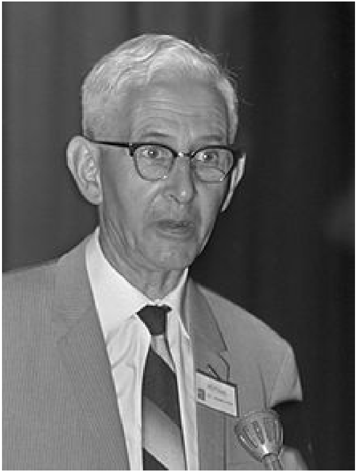
Heyting algebras



Arend Heyting

- An algebra of type $\langle A; \vee, \wedge, \to, 0, 1 \rangle$ such that:
  - $\langle A; \vee, \wedge, 0, 1 \rangle$ is a bounded lattice
  - $\to$ is a binary operation satisfying:

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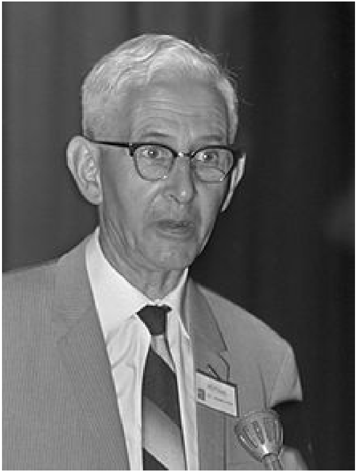
Heyting algebras



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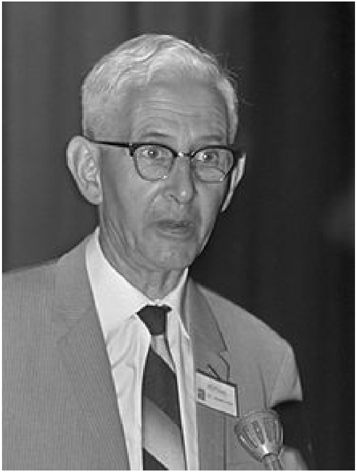
Heyting algebras



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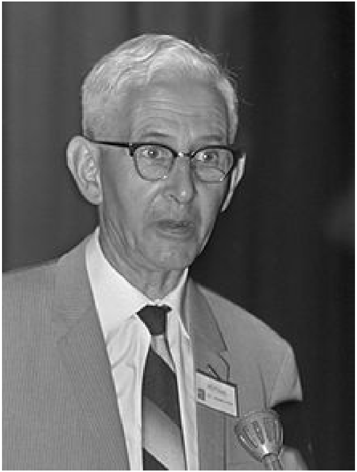

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- Quotients?
  - Groups: normal subgroups
  - Rings: ideals
  - In general: congruences

Filters

Definition

Let **L** be a lattice and let $F \subseteq L$. Then F is a *filter* provided that:

1. F is an upset, and,
2. if $x, y \in F$ then $x \wedge y \in F$.

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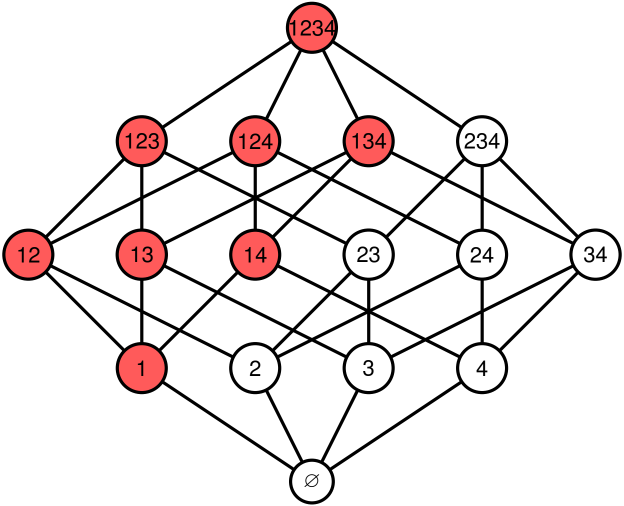
Fundamental theorem of Heyting algebras

Every congruence on a Heyting algebra arises from a filter.

Example



Expansions

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Theorem

Let F be a filter. Then F is compatible with a unary function f if and only if

$$x \leftrightarrow y \in F \implies fx \leftrightarrow fy \in F,$$

where $x \leftrightarrow y = x \to y \wedge y \to x$.

Not entirely convenient

- A filter F is compatible with $\langle A; \vee, \wedge, \rightarrow, f, g, h, 0, 1\rangle$ if $x \leftrightarrow y \in F$ implies:
  - $fx \leftrightarrow fy \in F$,
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Let **A** be an expanded Heyting algebra and let t be a unary term in the language of **A**. We say that t is a *normal filter term* on **A** if a filter is a normal filter if and only if F is closed under t.

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Example

The identity function is a normal filter term for Heyting algebras

Examples

Definition

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Theorem ("Folklore")

Let **A** *be a boolean algebra equipped with finitely many operators, denoted* $f_1, f_2, \ldots, f_n$. *Then the term* t, *defined by*

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Theorem (Sankappanavar, 1985)

Double-Heyting algebras possess a normal filter term.

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Lemma (Hasimoto, 2001)

Let $M = \{f_1, f_2, \ldots, f_n\}$ be a finite set of operators. Then $[M]$ exists, and

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Constructing normal filter terms

Definition

A unary map f is an *anti-operator* if $f(x \wedge y) = fx \vee fy$, and, $f1 = 0$. Let $\neg x$ be the unary term defined by $\neg x = x \to 0$.

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Example (Meskhi, 1982)

If $\mathbf{A}$ is a Heyting algebra with involution, i.e. a Heyting algebra equipped with a single unary operation i that is a dual automorphism. The map $tx := \neg ix$ is a normal filter term on $\mathbf{A}$.

The dual pseudocomplement

Example

Let **A** be an EHA. A unary operation $\sim$ is a *dual pseudocomplement operation* if the following equivalence is satisfied for all $x \in A$:

$$x \vee y = 1 \iff y \geq {\sim}x.$$

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Corollary (Sankappanavar, 1985)

Let **A** *be a dually pseudocomplemented Heyting algebra. Then* $\neg{\sim}$ *is a normal filter term on* **A***.*

Subdirectly irreducibles

Lemma

Let $\mathbf{A}$ *be an EHA, let* t *be a normal filter term on* $\mathbf{A}$*, and let* $dx = x \wedge tx$*.*

1. $\mathbf{A}$ *is subdirectly irreducible if and only if there exists* $b \in A \backslash \{1\}$ *such that for all* $x \in A \backslash \{1\}$ *there exists* $n \in \omega$ *such that* $d^n x \leq b$*.*
2. $\mathbf{A}$ *is simple if and only if for all* $x \in A \backslash \{1\}$ *there exists* $n \in \omega$ *such that* $d^n x = 0$*.*

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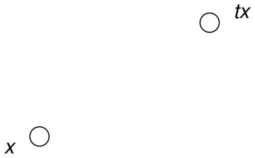

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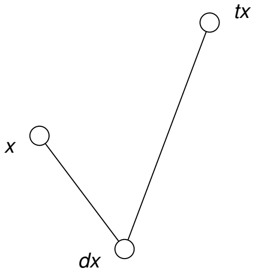

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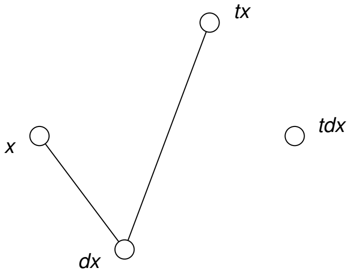

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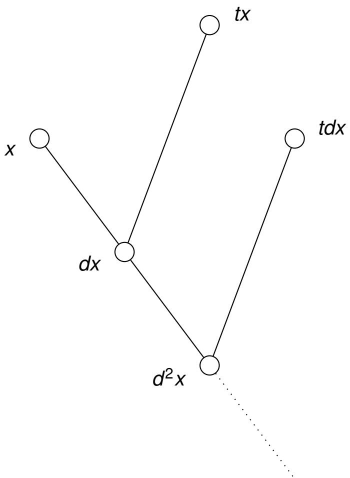

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Discriminator varieties

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A variety $\mathcal{V}$ has *definable principal congruences* (DPC) if you can define principal congruences by a first order formula. If this is done with a finite conjunction of equations then $\mathcal{V}$ has EDPC.

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Theorem (T., 2016)

Let $\mathcal{V}$ be a variety of EHAs with a common normal filter term t, and let $dx = x \wedge tx$. Then the following are equivalent:

1. *$\mathcal{V}$ has DPC,*
2. *$\mathcal{V}$ has EDPC,*
3. *$\mathcal{V} \models d^{n+1}x = d^n x$ for some $n \in \omega$.*

Discriminator varieties

Definition

- A variety is *semisimple* if every subdirectly irreducible algebra is simple.
- A variety is congruence permutable if for every pair of congruences θ_1, θ_2 on every algebra, $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$
- A variety is a discriminator variety if there is a ternary term t in the language of $\mathcal{V}$ such that t is a discriminator term on every subdirectly irreducible member of $\mathcal{V}$, i.e.,

$$t(x, y, z) = \begin{cases} x & \text{if } x \neq y \\ z & \text{if } x = y. \end{cases}$$

Discriminator varieties

Theorem (Blok, Köhler and Pigozzi, 1984)

Let $\mathcal{V}$ be a variety of any signature. The following are equivalent:

1. *$\mathcal{V}$ is semisimple, congruence permutable, and has EDPC.*
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Theorem (T., 2016)

Let $\mathcal{V}$ be a variety of dually pseudocomplemented EHAs and assume $\mathcal{V}$ has a normal filter term t. Then the following are equivalent.

1. *$\mathcal{V}$ is semisimple.*
2. *$\mathcal{V}$ is a discriminator variety.*

This generalises a result by Kowalski and Kracht (2006) for BAOs and a result of mine (2015) to appear for double-Heyting algebras.