Algebras of hypergraphs: representing regular double p-algebras

Christopher Taylor

La Trobe University

September 15, 2016

Constructing boolean algebras

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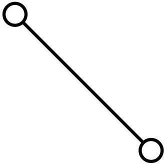

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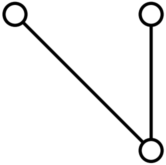

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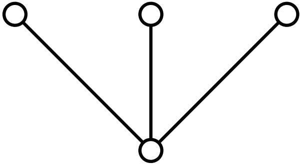

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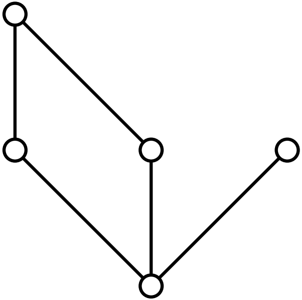

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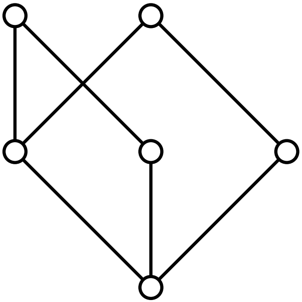

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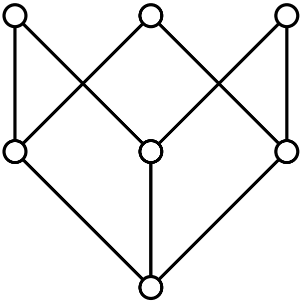

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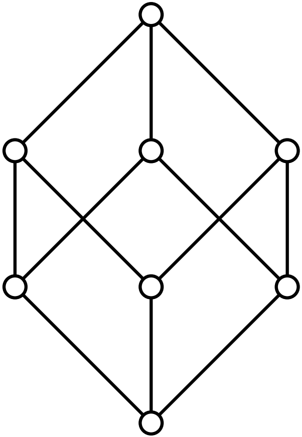

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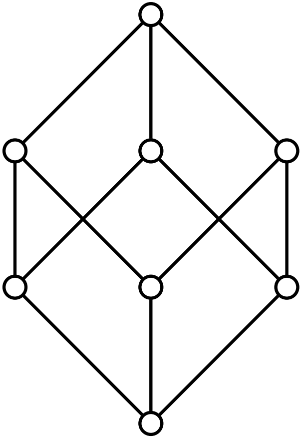

- $x \vee y = x \cup y$
- $x \wedge y = x \cap y$
- $\neg x = S \backslash x$

Finite characterisation

Theorem

Let L be a finite lattice. Then the following are equivalent.

1. *L is a boolean lattice,*
2. *$L \cong \mathcal{P}(B)$ for some finite set B.*

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Let $\mathrm{FC}(\mathbb{N})$ denote the set of finite or cofinite subsets of $\mathbb{N}$. It is easily assigned the structure of a boolean algebra, but is the wrong cardinality to come from a powerset.

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We will return to the infinite case later.

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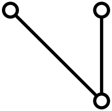

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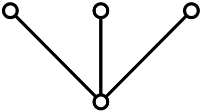

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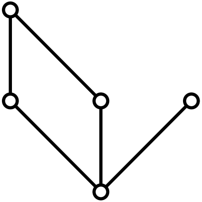

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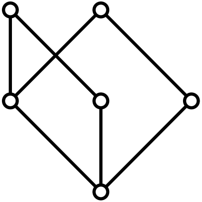

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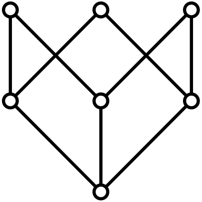

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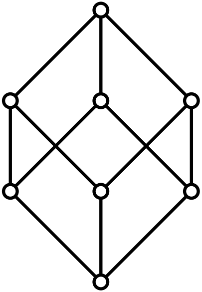

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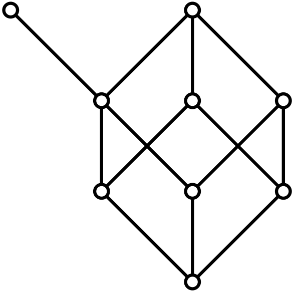

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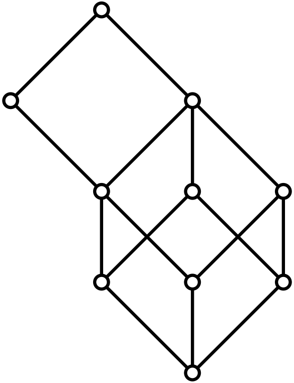

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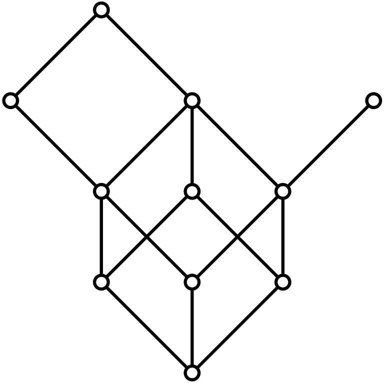

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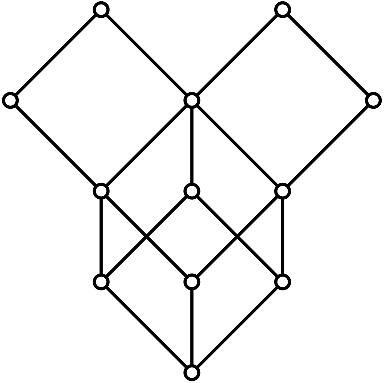

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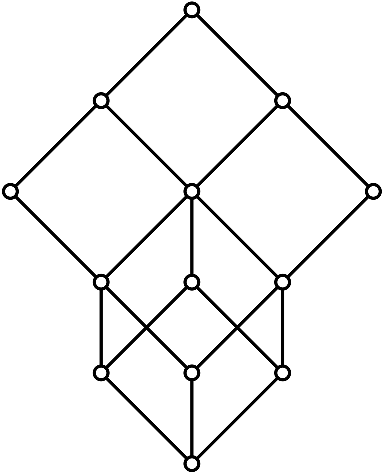

The lattice of subgraphs

- Let $G = \langle V, E \rangle$ be a graph. The set of all subgraphs of G induces a bounded distributive lattice, which we will call $\mathcal{S}(G)$, where

$$\langle V_1, E_1 \rangle \vee \langle V_2, E_2 \rangle = \langle V_1 \cup V_2, E_1 \cup E_2 \rangle$$
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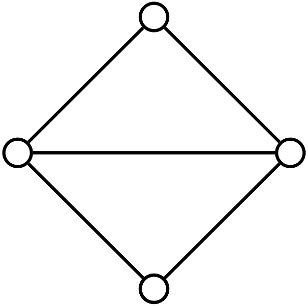
Let G be a graph. Then $\mathcal{S}(G)$ naturally forms a double-Heyting algebra.

Graph complements



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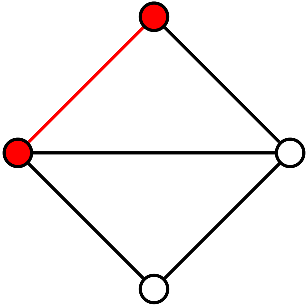

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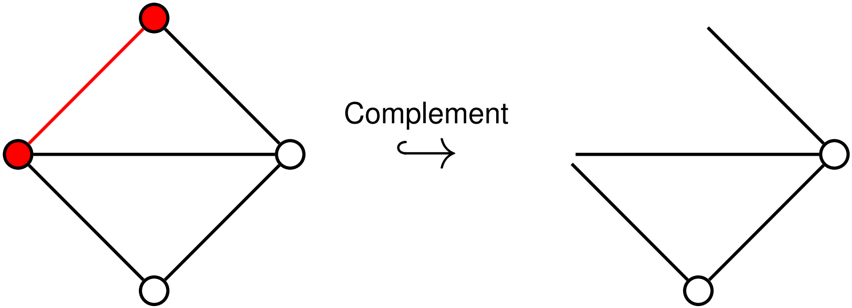

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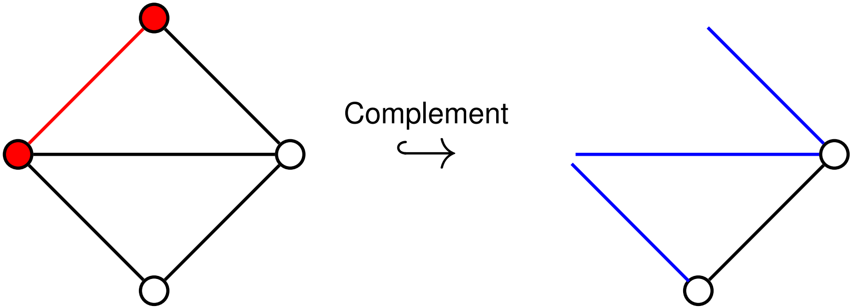

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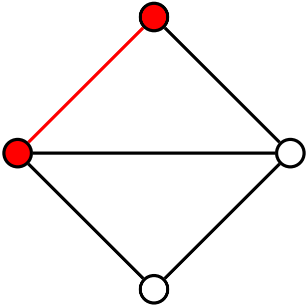

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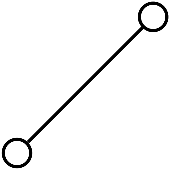

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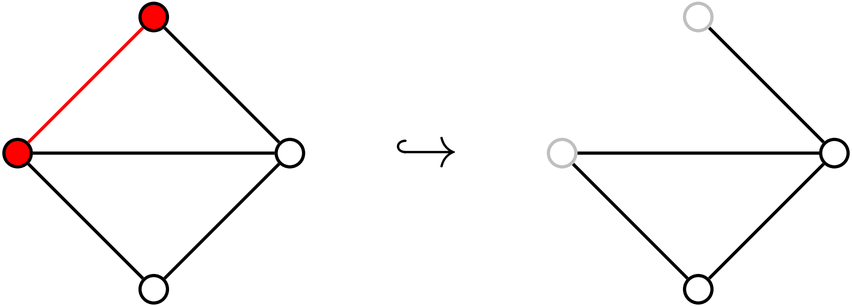

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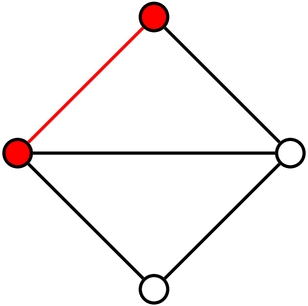

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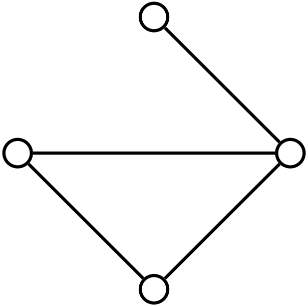

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Definition

An algebra $\mathbf{A} = \langle A; \vee, \wedge, \neg, \sim, 0, 1\rangle$ is a *double p-algebra* if $\langle A; \vee, \wedge, 0, 1\rangle$ is a bounded lattice, and $\neg$ and $\sim$ are the pseudocomplement and dual pseudocomplement respectively.

The algebra of subgraphs

Pseudocomplement

Take the set complement of the subgraph and abandon the extra edges. Formally, for a graph $G = \langle V, E \rangle$ and a subgraph $H = \langle V', E' \rangle$:

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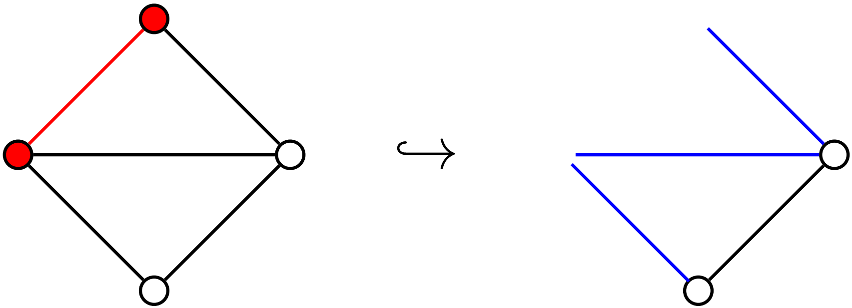

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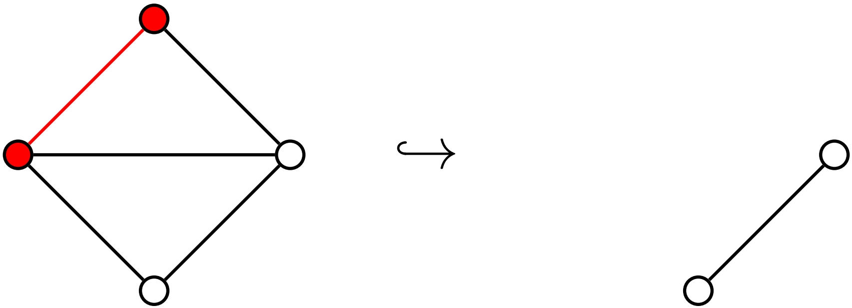

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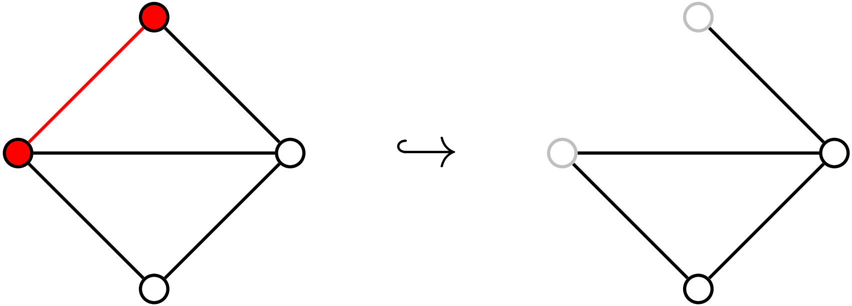

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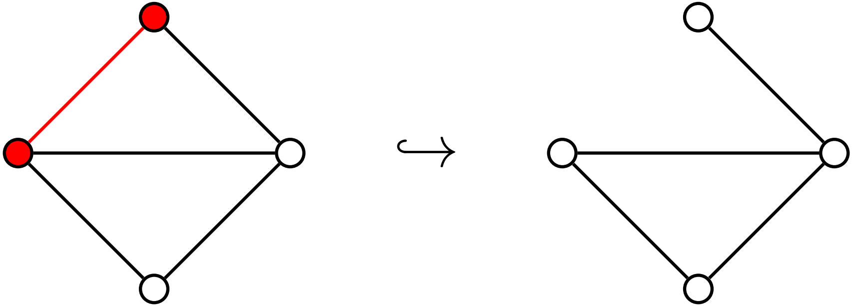

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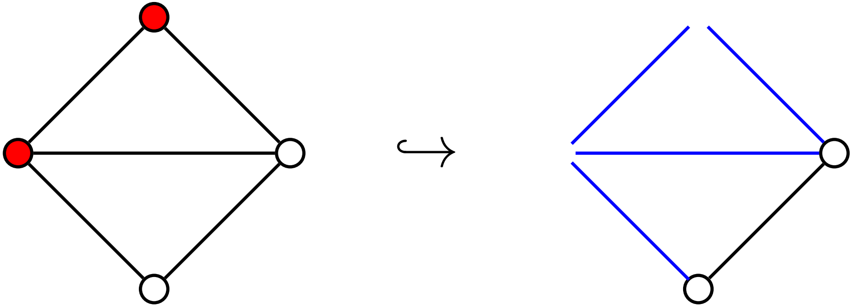

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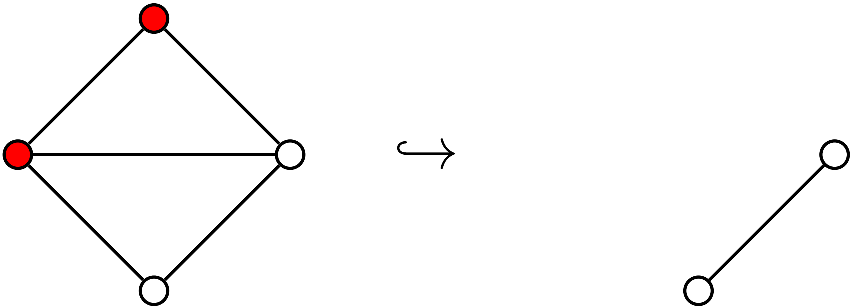

Regular double p-algebras

- Let **A** be an algebra. We say that **A** is *congruence regular* if, for all $\alpha, \beta \in \text{Con}(\mathbf{A})$, we have

$$((\exists x \in A)\ x/\alpha = x/\beta) \implies \alpha = \beta.$$

- Example: groups

Regular double p-algebras

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- Example: groups

Theorem (Varlet, 1972)

Let **A** *be a double p-algebra. Then the following are equivalent.*

1. **A** *is congruence regular.*
2. $(\forall a, b \in A)$ *if* $\neg a = \neg b$ *and* $\sim a = \sim b$ *then* $a = b$.
3. $(\forall a, b \in A)\ a \wedge \sim a \leq b \vee \neg b$.

A well-behaved structure

Theorem

Let $G = \langle V, E \rangle$ *be a graph. Then* $\mathcal{S}(G)$ *is (the underlying lattice of) a regular double p-algebra.*

A well-behaved structure

Theorem

Let $G = \langle V, E\rangle$ be a graph. Then $\mathcal{S}(G)$ is (the underlying lattice of) a regular double p-algebra.

Proof.

Let $A = \langle A_V, A_E\rangle$ and $B = \langle B_V, B_E\rangle$ be subgraphs of G. Recall that for a subgraph $H = \langle V', E'\rangle$,

$$\neg H = \langle V \backslash V', \{e \in E \backslash E' \mid (\forall x \in e)\ x \in V \backslash V'\}\rangle \quad (1)$$

$$\sim H = \langle V \backslash V' \cup \{v \in V \mid (\exists e \in E \backslash E')\ v \in e\}, E \backslash E'\rangle. \quad (2)$$

Assume $\neg A = \neg B$ and $\sim A = \sim B$. Then from (1) we have $V \backslash A_V = V \backslash B_V$ and from (2) we have $E \backslash A_E = E \backslash B_E$. Hence, $A = B$. □

Some results from the literature

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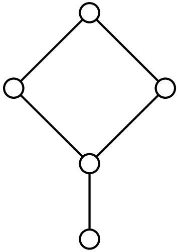
Theorem (Katriňák, 1973)

*Let **A** be a regular double p-algebra. Then **A** is term-equivalent to a double-Heyting algebra via the term*

$$x \to y = \neg\neg(\neg x \vee \neg\neg y) \wedge [\sim(x \vee \neg x) \vee \neg x \vee y \vee \neg y]$$

and its dual.

Are graphs enough?



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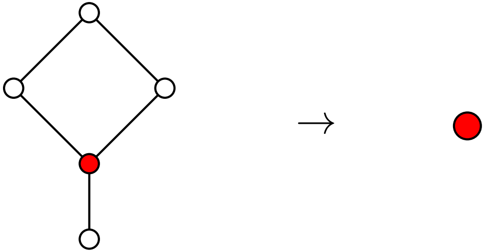

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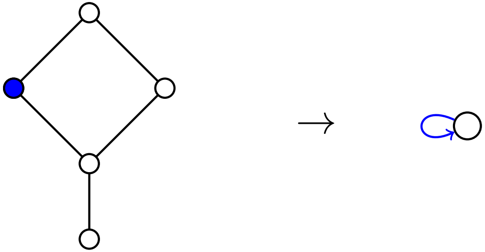

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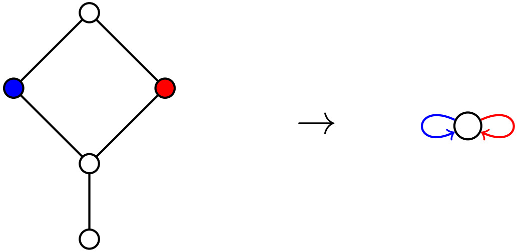

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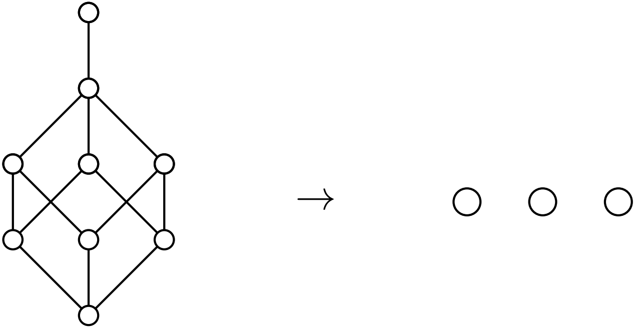

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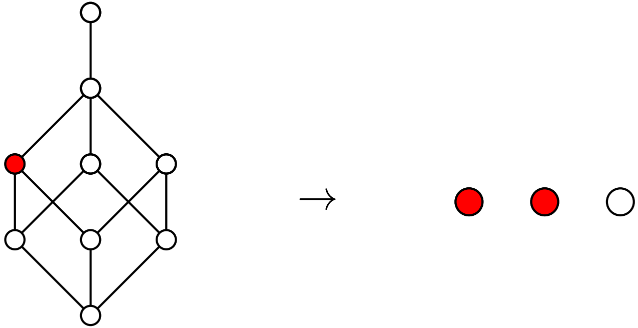

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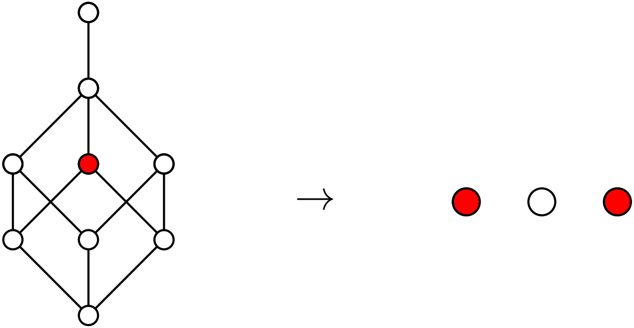

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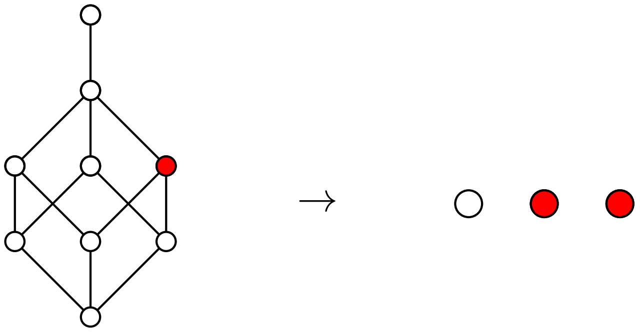

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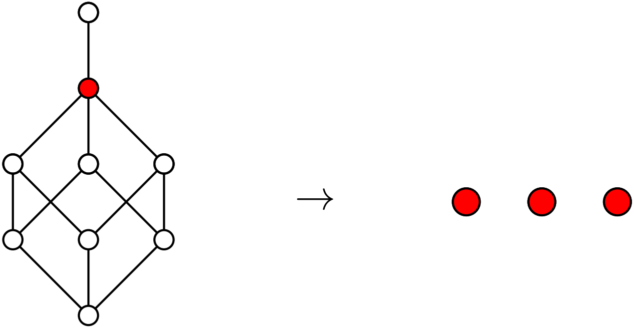

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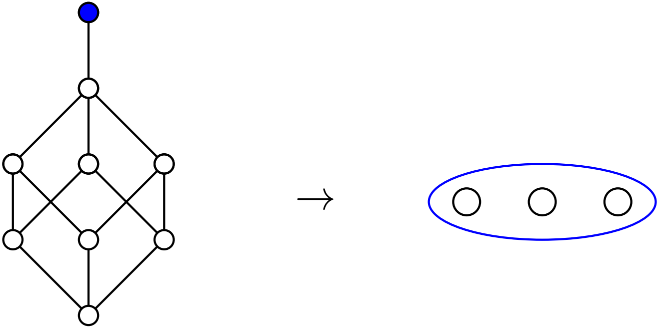

Hypergraphs

Definition

An *hypergraph* is a triple $\langle P, L, I\rangle$ where P is a set of points, L is a set of lines and $I \subseteq P \times L$ is an incidence relation describing which points are incident to which lines.

Example

Let $P = \{1, 2, 3\}$, $L = \{x, y, z, a, b\}$, and let

$$I = \{1,2,3\} \times \{x, y\} \\ \cup \{1,2\} \times \{z\} \\ \cup \{(1, a), (1, b)\}$$

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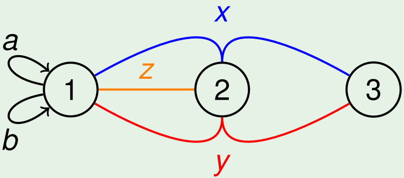

Subhypergraphs

Definition

Let $G = \langle P, L, I \rangle$ be a hypergraph. A *sub*(*hyper*)*graph* of G is a pair $\langle P', L' \rangle$ such that

1. $P' \subseteq P$ and $L' \subseteq L$,
2. for all $\ell \in L'$, if $(p, \ell) \in I$ then $p \in P'$.

The incidence relation is defined implicitly from G.

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Let $\mathcal{S}(G)$ denote the set of all subgraphs of a hypergraph G. This induces a double p-algebra in a similar way to graphs, where

$$\neg\langle P', L' \rangle = \langle P \setminus P', \{\ell \in L \setminus L' \mid (\forall p \in P)\ (p, \ell) \in I \implies p \in P \setminus P'\} \rangle,$$
$$\sim\langle P', L' \rangle = \langle P \setminus P' \cup \{p \in P \mid (\exists \ell \in L \setminus L')\ (p, \ell) \in I\}, L \setminus L' \rangle.$$

The main result (finite version)

Theorem

Let L be a finite lattice. Then the following are equivalent.

1. *L is a boolean lattice,*
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Theorem (T., 2015)

Let L be a finite lattice. Then the following are equivalent.

1. *L is (the underlying lattice of) a regular double p-algebra,*
2. $L \cong \mathcal{S}(G)$ *for some finite hypergraph G,*
3. $L \cong \mathcal{P}(B) \times \mathcal{S}(G)$ *for some finite set B and some finite hypergraph G.*

The proof begins

Lemma

Let $\{G_i \mid i \in I\}$ *be a set of mutually disjoint hypergraphs. Then*

$$\mathcal{S}(\bigcup_{i \in I} G_i) \cong \prod_{i \in I} \mathcal{S}(G_i).$$

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Define the map $\varphi \colon \mathcal{S}(\bigcup_{i \in I} G_i) \to \prod_{i \in I} \mathcal{S}(G_i)$ by $\varphi(H)(j) = H \cap G_j$. Then φ is the required isomorphism. □

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$L \cong \mathcal{S}(G)$ for some hypergraph G if and only if $L \cong \mathcal{P}(B) \times \mathcal{S}(G)$ for some set B and some hypergraph G.

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Observe that these results hold for the infinite case as well.

Lemma

Let $\mathbf{A}$ *be a finite regular double-p algebra. Then there is a (possibly trivial) boolean algebra* $\mathbf{B}$ *and a regular double p-algebra* $\mathbf{C}$ *such that* $\mathbf{A} \cong \mathbf{B} \times \mathbf{C}$*, and every atom in* $\mathbf{C}$ *is below every coatom in* $\mathbf{C}$*.*

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Proof.

(Sketch) Let $\mathcal{A}(\mathbf{A})$ denote the set of atoms in $\mathbf{A}$ and let $\mathcal{C}(\mathbf{A})$ denote the set of coatoms in $\mathbf{A}$. Let

$$X = \{a \in \mathcal{A}(\mathbf{A}) \mid (\exists c \in \mathcal{C}(\mathbf{A}))\ a \nleq c\},$$
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We can prove that $\bigvee X \in \mathrm{Cen}(\mathbf{A})$, where $\neg \bigvee X = \bigwedge Y$. The theory of distributive lattices then tells us that

$$\mathbf{A} \cong \downarrow\bigvee X \times \downarrow\bigwedge Y.$$

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The lattices ${\downarrow}\bigvee X$ and ${\downarrow}\bigwedge Y$ have the desired algebraic structure. □

The hypergraph construction

Lemma

Let $\mathbf{A}$ *be a finite regular double p-algebra and assume that for all* $a \in \mathcal{A}(\mathbf{A})$ *and all* $c \in \mathcal{C}(\mathbf{A})$ *we have* $a \leq c$. *Then, there exists a hypergraph* G *with no isolated points and no empty lines such that* $\mathbf{A} \cong \mathcal{S}(G)$.

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Proof.

The required hypergraph will be given by $G = \langle P, L, I \rangle$, where $P = \mathcal{A}(\mathbf{A})$, $L = \mathcal{C}(\mathbf{A})$ and $I = \{(a, c) \in P \times L \mid a \leq {\sim}c\}$.

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The required hypergraph will be given by $G = \langle P, L, I \rangle$, where $P = \mathcal{A}(\mathbf{A})$, $L = \mathcal{C}(\mathbf{A})$ and $I = \{(a, c) \in P \times L \mid a \leq {\sim}c\}$. The isomorphism is given by $\varphi \colon \mathcal{S}(G) \to A$ by:

$$\varphi : \langle P_H, L_H \rangle \mapsto \bigvee P_H \vee \bigvee \{{\sim}c \mid c \in L_H\}.$$

□

A simple lemma

Definition

Let **A** be a doubly atomic lattice, and let $\mathcal{A}(x) = \mathcal{A}(\mathbf{A}) \cap {\downarrow}x$ and $\mathcal{C}(x) = \mathcal{C}(\mathbf{A}) \cap {\uparrow}x$.

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A large portion of the proof in the previous slides relies on the following simple observation:

Lemma

Let **A** *be a complete, doubly atomic double p-algebra and let $x, y \in A$. Then,*

1. $\neg x = \neg[\bigvee \mathcal{A}(\mathbf{x})]$ *and* $\sim x = \sim[\bigwedge \mathcal{C}(x)]$.
2. *If* **A** *is regular, then $\mathcal{A}(x) = \mathcal{A}(y)$ and $\mathcal{C}(x) = \mathcal{C}(y)$ together imply $x = y$.*

The infinite case

Definition

Let **A** be a complete lattice. We say that **A** is *completely distributive* if, for any doubly indexed set $\{x_{i,j} \mid i \in I, j \in J\}$, we have

$$\bigwedge_{i\in I}\bigvee_{j\in J} x_{i,j} = \bigvee_{f\in F}\bigwedge_{i\in I} x_{i,f(i)},$$

where F is the set of all functions from I to J.

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$$\bigvee X \wedge \bigvee Y = \bigvee \{x \wedge y \mid x \in X, y \in Y\} \qquad \text{(JID)}$$

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If "finite" is replaced with "complete, atomic, coatomic and satisfies both (MID) and (JID)" then all of the previous results hold.

The main result

Theorem

Let B be a boolean lattice. Then the following are equivalent.

1. $B \cong \mathcal{P}(X)$ *for some set* X.
2. B *is complete and atomic.*
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Theorem (T., 2015)

Let $\mathbf{A}$ *be a regular double p-algebra. Then the following are equivalent.*

1. $\mathbf{A} \cong \mathcal{P}(B) \times \mathcal{S}(G)$ *for some set* B *and some hypergraph* G.
2. $\mathbf{A} \cong \mathcal{S}(G)$ *for some hypergraph* G.
3. $\mathbf{A}$ *is complete, completely distributive and doubly atomic.*
4. $\mathbf{A}$ *is complete, satisfies* (JID) *and* (MID), *and is doubly atomic.*

Embedding theorem

Theorem (Stone's Theorem)

Let $\mathbf{B}$ *be a boolean algebra. Then there is a set* X *such that* $\mathbf{B}$ *embeds into* $\mathcal{P}(X)$.

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This rides on the Priestley duality for distributive lattices.