Algebraic logic

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SEMS Research Workshop 2016

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- All men are mortal
- Socrates is a man
- Therefore, Socrates is mortal

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- (IF x is a man THEN x is mortal) AND (Socrates is a man)

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Classical logic is the study of propositions formed by the logical connectives $\wedge$ (AND), $\vee$ (OR) and $\neg$ (NOT).

Basic and Derived Argument Forms		
Name	**Sequent**	**Description**
Modus Ponens	$((p \to q) \land p) \vdash q$	If p then q; p; therefore q
Modus Tollens	$((p \to q) \land \neg q) \vdash \neg p$	If p then q; not q; therefore not p
Hypothetical Syllogism	$((p \to q) \land (q \to r)) \vdash (p \to r)$	If p then q; if q then r; therefore, if p then r
Disjunctive Syllogism	$((p \lor q) \land \neg p) \vdash q$	Either p or q, or both; not p; therefore, q
Constructive Dilemma	$((p \to q) \land (r \to s) \land (p \lor r)) \vdash (q \lor s)$	If p then q; and if r then s; but p or r; therefore q or s
Destructive Dilemma	$((p \to q) \land (r \to s) \land (\neg q \lor \neg s)) \vdash (\neg p \lor \neg r)$	If p then q; and if r then s; but not q or not s; therefore not p or not r
Bidirectional Dilemma	$((p \to q) \land (r \to s) \land (p \lor \neg s)) \vdash (q \lor \neg r)$	If p then q; and if r then s; but p or not s; therefore q or not r
Simplification	$(p \land q) \vdash p$	p and q are true; therefore p is true
Conjunction	$p, q \vdash (p \land q)$	p and q are true separately; therefore they are true conjointly
Addition	$p \vdash (p \lor q)$	p is true; therefore the disjunction (p or q) is true
Composition	$((p \to q) \land (p \to r)) \vdash (p \to (q \land r))$	If p then q; and if p then r; therefore if p is true then q and r are true
De Morgan's Theorem (1)	$\neg(p \land q) \vdash (\neg p \lor \neg q)$	The negation of (p and q) is equiv. to (not p or not q)
De Morgan's Theorem (2)	$\neg(p \lor q) \vdash (\neg p \land \neg q)$	The negation of (p or q) is equiv. to (not p and not q)
Commutation (1)	$(p \lor q) \vdash (q \lor p)$	(p or q) is equiv. to (q or p)
Commutation (2)	$(p \land q) \vdash (q \land p)$	(p and q) is equiv. to (q and p)
Commutation (3)	$(p \leftrightarrow q) \vdash (q \leftrightarrow p)$	(p is equiv. to q) is equiv. to (q is equiv. to p)
Association (1)	$(p \lor (q \lor r)) \vdash ((p \lor q) \lor r)$	p or (q or r) is equiv. to (p or q) or r
Association (2)	$(p \land (q \land r)) \vdash ((p \land q) \land r)$	p and (q and r) is equiv. to (p and q) and r
Distribution (1)	$(p \land (q \lor r)) \vdash ((p \land q) \lor (p \land r))$	p and (q or r) is equiv. to (p and q) or (p and r)
Distribution (2)	$(p \lor (q \land r)) \vdash ((p \lor q) \land (p \lor r))$	p or (q and r) is equiv. to (p or q) and (p or r)
Double Negation	$p \vdash \neg\neg p$	p is equivalent to the negation of not p
Transposition	$(p \to q) \vdash (\neg q \to \neg p)$	If p then q is equiv. to if not q then not p
Material Implication	$(p \to q) \vdash (\neg p \lor q)$	If p then q is equiv. to not p or q
Material Equivalence (1)	$(p \leftrightarrow q) \vdash ((p \to q) \land (q \to p))$	(p iff q) is equiv. to (if p is true then q is true) and (if q is true then p is true)
Material Equivalence (2)	$(p \leftrightarrow q) \vdash ((p \land q) \lor (\neg p \land \neg q))$	(p iff q) is equiv. to either (p and q are true) or (both p and q are false)
Material Equivalence (3)	$(p \leftrightarrow q) \vdash ((p \lor \neg q) \land (\neg p \lor q))$	(p iff q) is equiv to., both (p or not q is true) and (not p or q is true)
Exportation[9]	$((p \land q) \to r) \vdash (p \to (q \to r))$	from (if p and q are true then r is true) we can prove (if q is true then r is true, if p is true)
Importation	$(p \to (q \to r)) \vdash ((p \land q) \to r)$	If p then (if q then r) is equivalent to if p and q then r
Tautology (1)	$p \vdash (p \lor p)$	p is true is equiv. to p is true or p is true
Tautology (2)	$p \vdash (p \land p)$	p is true is equiv. to p is true and p is true
Tertium non datur (Law of Excluded Middle)	$\vdash (p \lor \neg p)$	p or not p is true
Law of Non-Contradiction	$\vdash \neg(p \land \neg p)$	p and not p is false, is a true statement

More important laws

De Morgan's laws

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- NOT[have(cake)] OR NOT[eat(cake)]

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Every function $f : \{0,1\}^n \rightarrow \{0,1\}$ can be defined as a formula using the language of Boolean algebra.

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Theorem

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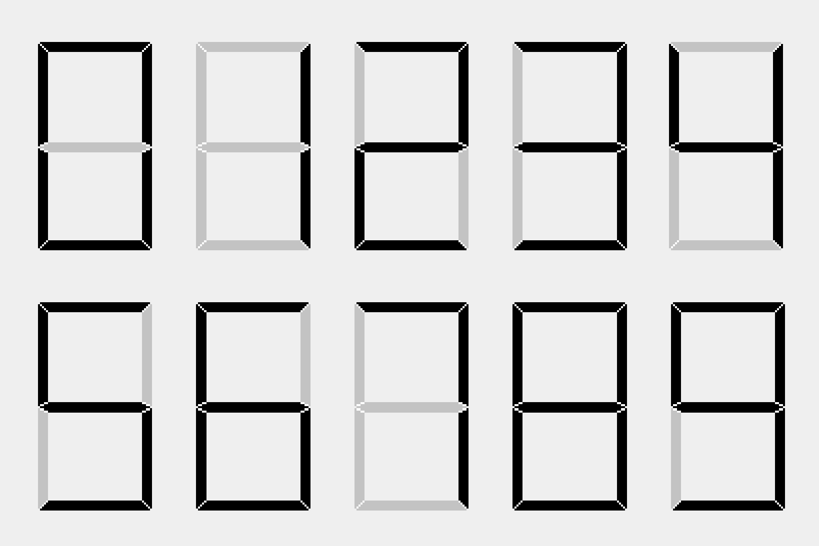
- Incredibly important for building logic circuits

In action



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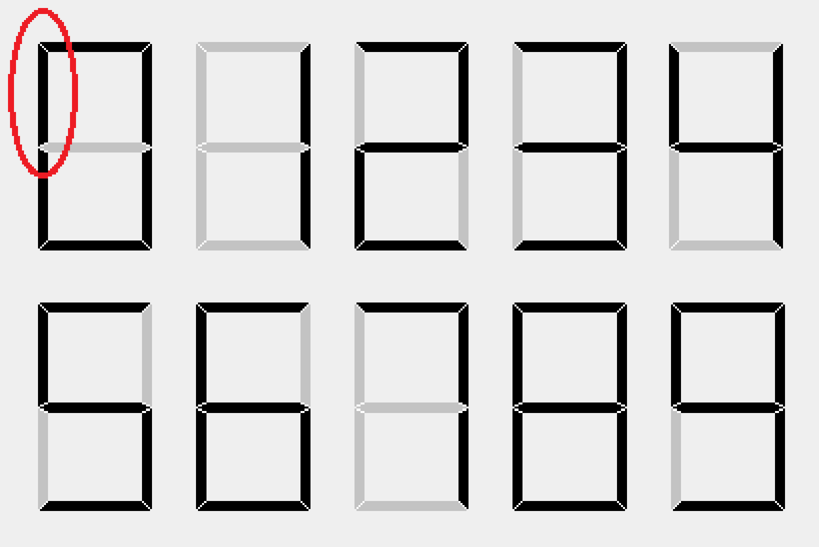

The function

number	binary	output?
0	0000	1
1	0001	0
2	0010	0
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A set: $\{a, b, c, 1, 2, 3\}$
Some subsets: $\{b\}; \{a, 1, 2\}; \{1, 3\}$

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And NOT like complements:

$$\text{NOT}(\{a, 1, 2\}) = \{a, b, c, 1, 2, 3\} - \{a, 1, 2\} = \{b, c, 3\}$$

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- Taylor, C. J.: Algebras of incidence structures: representations of regular double p-algebras. *Algebra universalis* (2016)