Heyting algebras with operators

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Heyting algebras

Definition

A *Heyting algebra* is a bounded distributive lattice equipped with a binary operation $\to$ satisfying the following equivalence:

$$x \wedge y \leq z \iff y \leq x \to z.$$

Just as boolean algebras arise from classical logic, Heyting algebras form the algebraic counterpart to intuitionistic logic.

Theorem

The class of Heyting algebras is an equational class, defined by the equations for bounded distributive lattices, along with:

1. $x \wedge (x \to y) = x \wedge y,$
2. $x \wedge (y \to z) = x \wedge [(x \wedge y) \to (x \wedge z)]$, *and,*
3. $(x \wedge y) \to x = 1.$

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Filters

Definition

Let **L** be a lattice and let $F \subseteq L$. Then F is a *filter* provided that:

1. F is an upset, and,
2. if $x, y \in F$ then $x \wedge y \in F$.

Definition

Let **A** be a Heyting algebra and let F be a filter on **A**. For all $x, y \in A$, let $x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$. Let $\theta(F)$ denote the relation given by

$$\theta(F) := \{(x, y) \mid x \leftrightarrow y \in F\}.$$

Theorem

Let **A** *be a Heyting algebra and let* F *be a filter on* **A**. *Then* $\theta(F)$ *is a congruence on* **A**, *and the map* $\theta\colon \mathrm{Fil}(\mathbf{A}) \to \mathrm{Con}(\mathbf{A})$ *is an isomorphism, with the inverse given by the map* $\alpha \mapsto 1/\alpha$.

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Normal filters

Definition

Let **A** be a Heyting algebra, let $f\colon A^n \to A$ be any map and let F be a filter on **A**. We say that F is *normal with respect to f* if, for all $x_1, y_1, \ldots, x_n, y_n \in A$,

$$\{x_i \leftrightarrow y_i \mid i \leq n\} \subseteq F \implies f(x_1, \ldots, x_n) \leftrightarrow f(y_1, \ldots, y_n) \in F.$$

Example

If f is unary, then F is normal with respect to f provided that, for all $x, y \in A$, if $x \leftrightarrow y \in F$ then $fx \leftrightarrow fy \in F$.

Definition

Let M be a set of operations on A and let F be a filter on **A**. We say that F is *normal with respect to M* if it is normal with respect to f for every $f \in M$.

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Expanded Heyting algebras

Definition

Let $\mathbf{A} = \langle A; M, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is an *expanded Heyting algebra* (EHA) if the reduct $\langle A; \vee, \wedge, \rightarrow, 0, 1 \rangle$ is a Heyting algebra and M is a set of operations on A.

The set M will remain fixed but arbitrary throughout this talk. Recall that for any filter F on $\mathbf{A}$, the congruence $\theta(F)$ is defined by

$$\theta(F) := \{(x, y) \mid x \leftrightarrow y \in F\},$$

and F is normal with respect to a (unary) map f if,

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Let $\mathbf{A}$ be an EHA and let F be a filter on $\mathbf{A}$. Then $\theta(F)$ is a congruence on $\mathbf{A}$ if and only if F is normal with respect to M.

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From now on, any unquantified **A** will be a fixed but arbitrary EHA.

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We will say that a filter F on **A** is a *normal filter on* **A** if it is normal with respect to M. Let $\mathbf{Fil}(\mathbf{A})$ denote the lattice of normal filters on **A** and let $\mathrm{Fg}^{\mathbf{A}}(a)$ denote the normal filter generated by a.

Definition

Let t be a unary term in the language of **A**. We say that t is a *normal filter term* (*on* **A**) provided that, for all $x, y \in A$ and every filter F on **A**:

1. if $x \leq y$ then $t^{\mathbf{A}}x \leq t^{\mathbf{A}}y$, and,
2. F is a normal filter if and only if F is closed under $t^{\mathbf{A}}$.

Henceforth we will drop the superscripts on term functions.

Example

The identity function is a normal filter term for Heyting algebras.

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Boolean algebras with operators

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Let **A** be a bounded lattice and let f be a unary operation on A. The map f is a (dual normal) *operator* if $f(x \wedge y) = fx \wedge fy$, and, $f1 = 1$.

Definition

An algebra $\mathbf{A} = \langle A; \{f_i \mid i \in I\}, \vee, \wedge, \neg, 0, 1\rangle$ is a *boolean algebra with operators* (BAO) if $\langle A; \vee, \wedge, \neg, 0, 1\rangle$ is a boolean algebra and each f_i is an operator.

Theorem (Folklore)

*Let **A** be a BAO of finite type. Then the term t, defined by*

$$tx = \bigwedge\{f_i x \mid i \in I\}$$

*is a normal filter term on **A**.*

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Heyting algebras with operators

Hasimoto[1] developed a construction which produces a partial unary operation that plays some role in defining congruences. We let $[M]$ denote the result of his construction. The construction does not apply in general, so we say that $[M]$ *exists* if it does.

Lemma (Hasimoto, 2001)

Let $\mathbf{A}$ *be an EHA of finite type and assume every operation in* M *is an operator. Then* $[M]$ *exists, and*

$$[M]x = \bigwedge\{f_i x \mid i \in I\}.$$

With the definition, it is not be hard to prove that:

1. If M is finite and $[\{f\}]$ exists for each $f \in M$, then $[M]$ exists, and,
2. If $[M]$ is term-definable in the language of $\mathbf{A}$, it guarantees a normal filter term.

[1] Hasimoto, Y., *Heyting algebras with operators*, MLQ Math. Log. Q., 47 (2001), 2, 187–196.

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Let $\mathbf{A}$ be a Heyting algebra and let f be a unary operation on A. The map f is an *anti-operator* if $f(x \wedge y) = fx \vee fy$, and, $f1 = 0$. Let $\neg x$ be the unary term defined by $\neg x = x \to 0$.

Lemma (T., 2016)

Let $\mathbf{A}$ be an EHA and let f be an anti-operator on A. Then $[f]$ exists, and

$$[f]x = \neg fx$$

Example

Let $\mathbf{A}$ be an EHA. A unary operation $\sim$ is a *dual pseudocomplement operation* if the following equivalence is satisfied for all $x \in A$:

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A *dually pseudocomplemented Heyting algebra* is an EHA with $M = \{\sim\}$.

Corollary (Sankappanavar[1], 1985)

Let $\mathbf{A}$ *be a dually pseudocomplemented Heyting algebra. Then* $\neg\sim$ *is a normal filter term on* $\mathbf{A}$.

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Some consequences

Lemma

Let $\mathbf{A}$ *be an EHA, let* t *be a normal filter term on* $\mathbf{A}$*, and let* $dx = x \wedge tx$*.*

1. $\mathrm{Fg}^{\mathbf{A}}(x) = \bigcup_{n \in \omega} {\uparrow} d^n x$.
2. $(y, 1) \in \mathrm{Cg}^{\mathbf{A}}(x, 1)$ *if and only if* $y \geq d^n x$ *for some* $n \in \omega$*.*

Lemma

Let $\mathbf{A}$ *be an EHA, let* t *be a normal filter term on* $\mathbf{A}$*, and let* $dx = x \wedge tx$*.*

1. $\mathbf{A}$ *is subdirectly irreducible if and only if there exists* $b \in A \backslash \{1\}$ *such that for all* $x \in A \backslash \{1\}$ *there exists* $n \in \omega$ *such that* $d^n x \leq b$*.*
2. $\mathbf{A}$ *is simple if and only if for all* $x \in A \backslash \{1\}$ *there exists* $n \in \omega$ *such that* $d^n x = 0$*.*

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EDPC

Definition

A variety $\mathcal{V}$ has *definable principal congruences* (DPC) if there exists a first-order formula $\varphi(x, y, z, w)$ in the language of $\mathcal{V}$ such that, for all $\mathbf{A} \in \mathcal{V}$, and all $a, b, c, d \in A$, we have

$$(a, b) \in \mathrm{Cg}^{\mathbf{A}}(c, d) \iff \mathbf{A} \models \varphi(a, b, c, d).$$

If φ is a finite conjunction of equations then $\mathcal{V}$ has *equationally definable principal congruences* (EDPC).

Theorem (T., 2016)

Let $\mathcal{V}$ be a variety of EHAs with a common normal filter term t, and let $dx = x \wedge tx$. Then the following are equivalent:

1. *$\mathcal{V}$ has EDPC,*
2. *$\mathcal{V}$ has DPC,*
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Proof.

We will prove that if $\mathcal{V}$ has DPC then there exists $n \in \omega$ such that $\mathcal{V} \models d^{n+1}x = d^n x$. Suppose otherwise. Then, for each $i \in \omega$ there exists $\mathbf{A}_i \in \mathcal{V}$ and $a_i \in A_i$ such that $d^i a_i \neq d^{i+1} a_i$. Let $\mathbf{A}$ be the product

$$\mathbf{A} := \prod_{i \in \omega} \mathbf{A}_i.$$

Now let $a = \langle a_i \mid i \in \omega \rangle$, and let $b = \langle d^{i+1} a_i \mid i \in \omega \rangle$. For each $i \in \omega$ we have $(d^{i+1} a_i, 1_i) \in \mathrm{Cg}^{\mathbf{A}_i}(a_i, 1_i)$, and so $\mathbf{A}_i \models \varphi(d^{i+1} a_i, 1_i, a_i, 1_i)$. It then follows that $\mathbf{A} \models \varphi(b, 1, a, 1)$ and so $(b, 1) \in \mathrm{Cg}^{\mathbf{A}}(a, 1)$. But now, from an earlier lemma, it follows that there exists $k \in \omega$ such that $b \geq d^k a$. In particular, then, we have

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Discriminator varieties

Definition

A variety is *semisimple* if every subdirectly irreducible member of $\mathcal{V}$ is simple. If there is a ternary term t in the language of $\mathcal{V}$ such that t is a discriminator term on every subdirectly irreducible member of $\mathcal{V}$, i.e.,

$$t(x, y, z) = \begin{cases} x & \text{if } x \neq y \\ z & \text{if } x = y, \end{cases}$$

then $\mathcal{V}$ is a *discriminator variety*.

Theorem (Blok, Köhler and Pigozzi[1])

Let $\mathcal{V}$ be a variety of any signature. The following are equivalent:

1. *$\mathcal{V}$ is semisimple, congruence permutable, and has EDPC.*
2. *$\mathcal{V}$ is a discriminator variety.*

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A main result

Theorem (T., 2016)

Let $\mathcal{V}$ be a variety of dually pseudocomplemented EHAs, assume $\mathcal{V}$ has a normal filter term t, and let $dx = \neg{\sim}x \wedge tx$. Then the following are equivalent.

1. $\mathcal{V}$ *is semisimple.*
2. $\mathcal{V}$ *is a discriminator variety.*
3. $\mathcal{V}$ *has DPC and there exists* $m \in \omega$ *such that* $\mathcal{V} \models x \leq d{\sim}d^m\neg x$.
4. $\mathcal{V}$ *has EDPC and there exists* $m \in \omega$ *such that* $\mathcal{V} \models x \leq d{\sim}d^m\neg x$.
5. *There exists* $n \in \omega$ *such that* $\mathcal{V} \models d^{n+1}x = d^n x$ *and* $\mathcal{V} \models d{\sim}d^n x = {\sim}d^n x$.

This generalises a result by Kowalski and Kracht[1] for BAOs and another result to appear for double-Heyting algebras[2]

[1] Kowalski, T., and M. Kracht, 'Semisimple varieties of modal algebras', Studia Logica, 83 (2006), 1-3, 351–363.

[2] Taylor, C., 'Discriminator varieties of double-Heyting algebras', Rep. Math. Logic, (2016). To appear.

Constructing normal filter terms

Definition

Let **A** be a Heyting algebra and let $f\colon A^n \to A$ be a map. For each $a \in A$, define the set $f^{\leftrightarrow}(a)$ by

$$f^{\leftrightarrow}(a) = \{f(x_1, \ldots, x_n) \leftrightarrow f(y_1, \ldots, y_n) \mid x_i \leftrightarrow y_i \geq a\}$$

Informally, the set $f^{\leftrightarrow}(a)$ is a set of elements that "should" be in a normal filter containing a if f is in the signature of **A**. If the infimum of $f^{\leftrightarrow}(a)$ exists, then that element encapsulates some of this information.

Definition

For any set K of maps on A, let $[K]\colon A \to A$ be the partial operation

$$[K]a = \bigwedge \bigcup \{f^{\leftrightarrow}(a) \mid f \in K\}.$$

We say that $[K]$ exists in **A** if it is defined for all $a \in A$. If $K = \{f\}$ we will write $[f]$ instead.

Constructing normal filter terms

The construction on the previous slide is due to Hasimoto[3].

Definition

Assume $[M]$ exists in $\mathbf{A}$. Let $\mathbf{A}^\flat$ denote the algebra

$$\langle A; \vee, \wedge, \rightarrow, [M], 0, 1 \rangle.$$

Theorem (Hasimoto, 2001)

Let $\mathbf{A}$ *be an EHA and assume* $[M]$ *exists in* $\mathbf{A}$*.*

1. $\mathrm{Fil}(\mathbf{A}^\flat) \subseteq \mathrm{Fil}(\mathbf{A})$ *and* $\mathrm{Con}(\mathbf{A}^\flat) \subseteq \mathrm{Con}(\mathbf{A})$*.*
2. *The following are equivalent:*
    1. $[M]a \in \mathrm{Fg}^{\mathbf{A}}(a)$ *for all* $a \in A$*,*
    2. $\mathbf{Fil}(\mathbf{A}) = \mathbf{Fil}(\mathbf{A}^\flat)$*, and,*
    3. $\mathbf{Con}(\mathbf{A}) = \mathbf{Con}(\mathbf{A}^\flat)$*.*

[3] Hasimoto, Y., *Heyting algebras with operators*, MLQ Math. Log. Q., 47 (2001), 2, 187–196.

Constructing normal filter terms

Lemma (Hasimoto, 2001)

Assume $[M]$ *exists in* $\mathbf{A}$*. Then, for all* $x, y \in A$*,*

1. $[M](x \wedge y) = [M]x \wedge [M]y$*, and,*
2. $[M]1 = 1$*.*

Lemma

Assume $[M]$ *exists in* $\mathbf{A}$*. If there is a unary term* t *in the language of* $\mathbf{A}$ *such that* $tx = [M]x$*, then* t *is a normal filter term on* $\mathbf{A}$*.*

Proof.

Let F be a filter on $\mathbf{A}$. If F is closed under t then by the previous theorem it is a normal filter on $\mathbf{A}$. Conversely, if F is a normal filter on $\mathbf{A}$, then whenever $(x, 1) \in \theta(F)$ we have $(tx, t1) \in \theta(F)$. But $t1 = 1$, and so $tx \in 1/\theta(F) = F$. □