Algebras of incidence structures: representing regular double p-algebras

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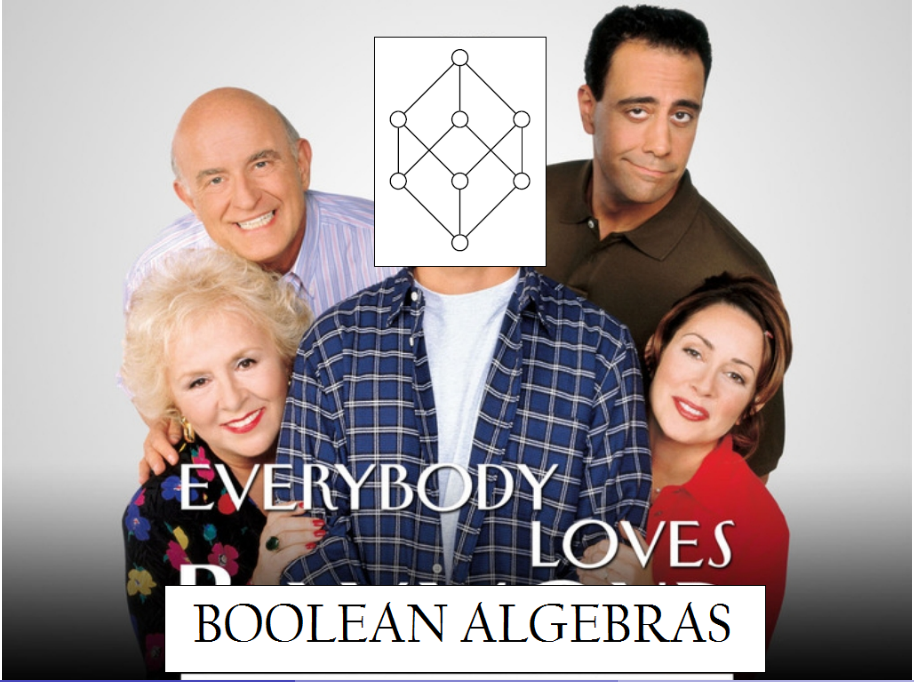
Victorian Algebra Conference 2015


EVERYBODY
LOVES
BOOLEAN ALGEBRAS

Boolean lattices

Theorem

Let L be a finite lattice. Then the following are equivalent.

1. *L is a boolean lattice,*
2. *$L \cong \mathcal{P}(B)$ for some finite set B,*
3. *$L \cong \mathbf{2}^n$ for some $n \geq 0$.*

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Theorem

Let B be a boolean lattice. Then the following are equivalent.

1. *$B \cong \mathcal{P}(X)$ for some set X.*
2. *B is complete and atomic.*
3. *B is complete and completely distributive.*

Some other classifications

- Birkhoff's duality for finite distributive lattices
- Stone's duality for boolean algebras
- Priestley's duality for bounded distributive lattices
- Every finite cyclic group is isomorphic to $\mathbb{Z}_n$ for some $n \in \omega$
- Every finite abelian group is isomorphic to $\prod_{i=0}^{n} \mathbb{Z}_{q_i}$ where each q_i is a power of a prime

Graphs

A graph:

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A subgraph:

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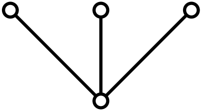

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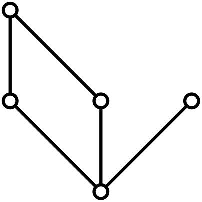

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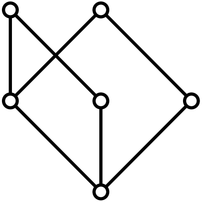

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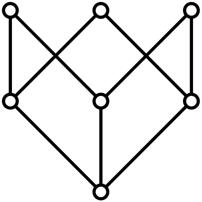

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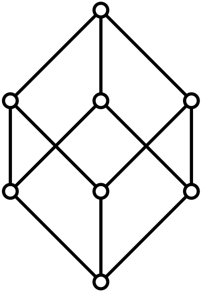

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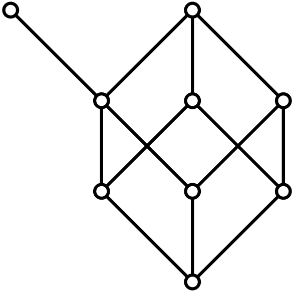

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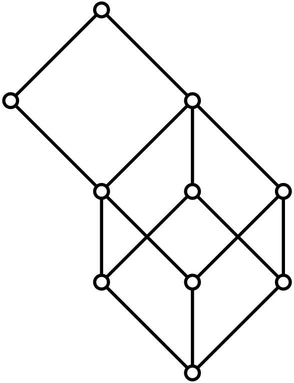

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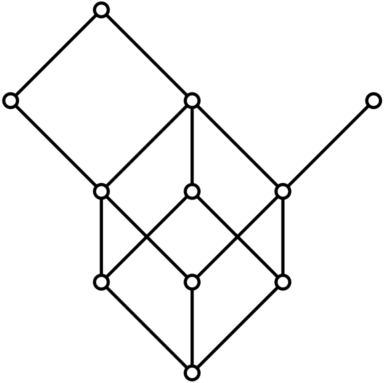

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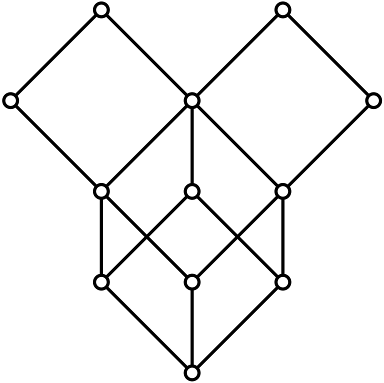

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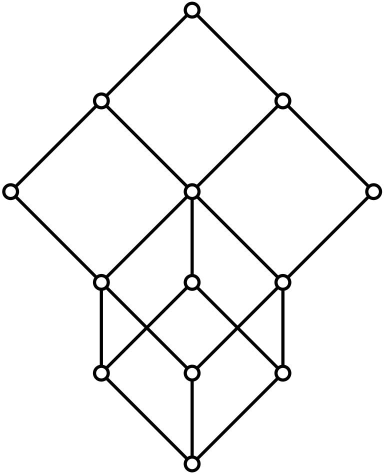

The lattice of subgraphs

- Let $G = \langle V, E\rangle$ be a graph. The set of all subgraphs of G induces a bounded distributive lattice, which we will call $\mathcal{S}(G)$, where

$$\langle V_1, E_1\rangle \vee \langle V_2, E_2\rangle = \langle V_1 \cup V_2, E_1 \cup E_2\rangle$$
$$\langle V_1, E_1\rangle \wedge \langle V_2, E_2\rangle = \langle V_1 \cap V_2, E_1 \cap E_2\rangle.$$

- Note that we permit the empty graph.

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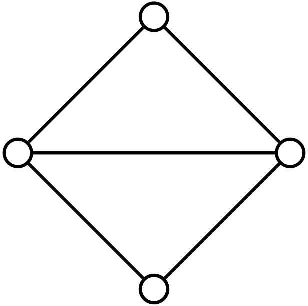
Theorem (Reyes & Zolfaghari, 1996)

Let G be a graph. Then $\mathcal{S}(G)$ naturally forms a double-Heyting algebra.

Graph complements



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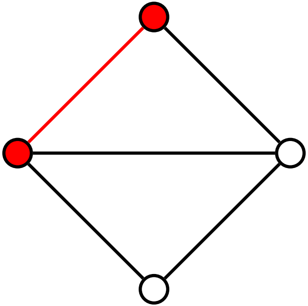

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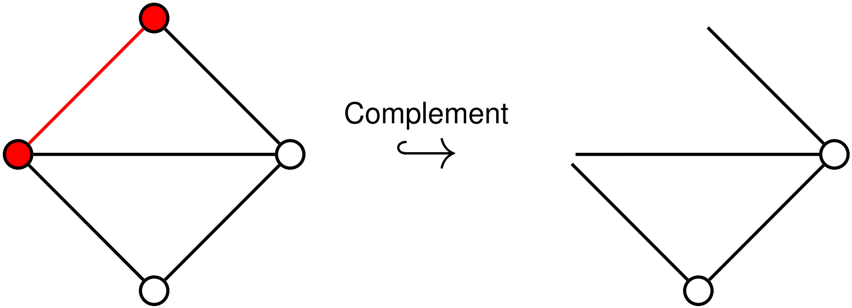

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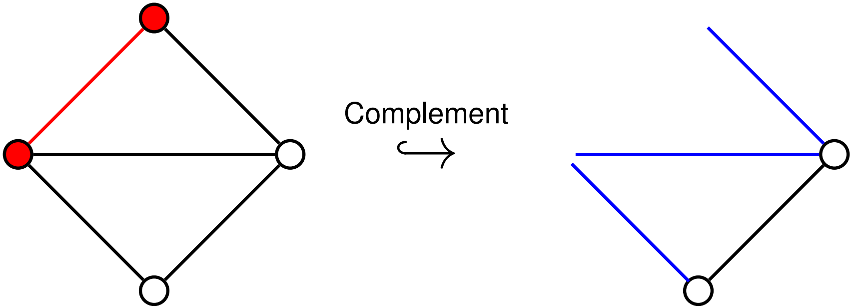

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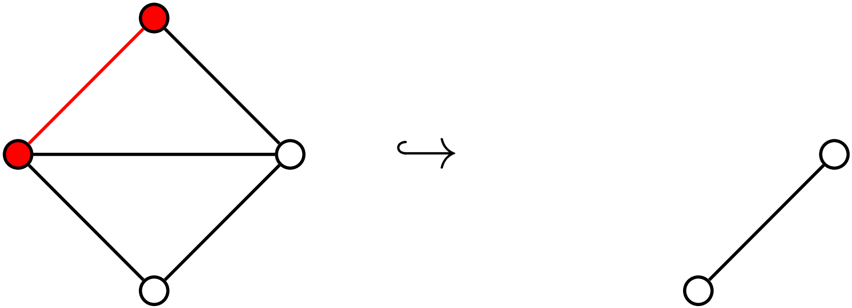

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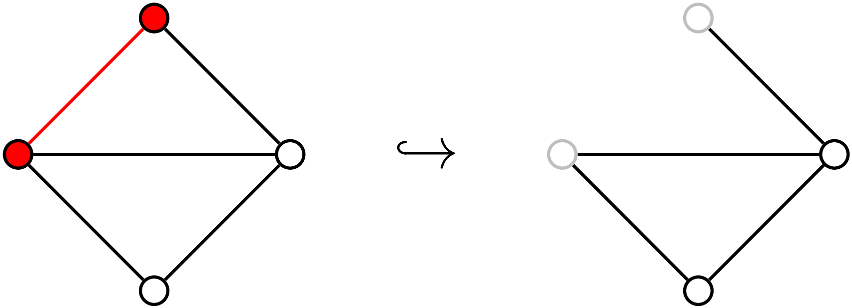

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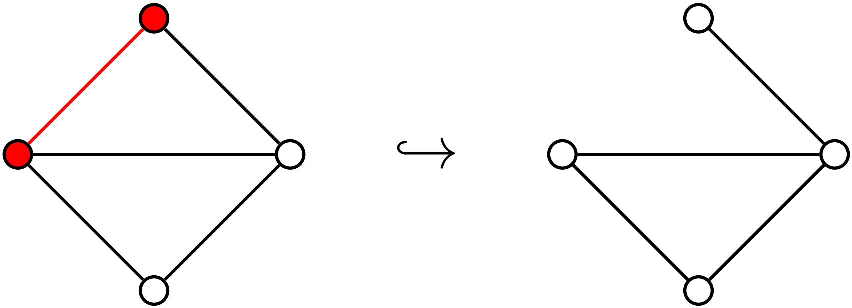

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Definition

An algebra $\mathbf{A} = \langle A; \vee, \wedge, 0, 1, *, + \rangle$ is a *double p-algebra* if $\langle A; \vee, \wedge, 0, 1 \rangle$ is a bounded lattice, and * and $^+$ are the pseudocomplement and dual pseudocomplement respectively.

The algebra of subgraphs

Pseudocomplement

Take the set complement of the subgraph and abandon the extra edges. Formally, for a graph $G = \langle V, E \rangle$ and a subgraph $H = \langle V', E' \rangle$:

$$H^* = \langle V \backslash V', \{e \in E \backslash E' \mid (\forall x \in e)\ x \in V \backslash V'\} \rangle$$

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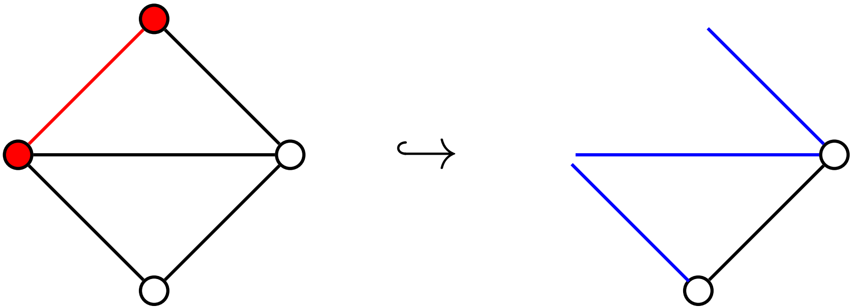

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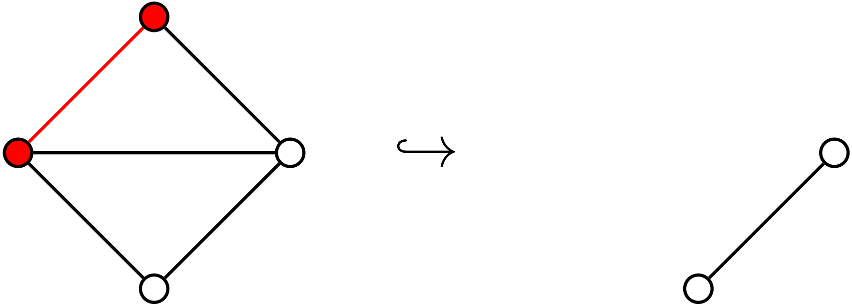

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Dual pseudocomplement

Just add the missing vertices back. Formally, for a graph $G = \langle V, E \rangle$ and a subgraph $H = \langle V', E' \rangle$:

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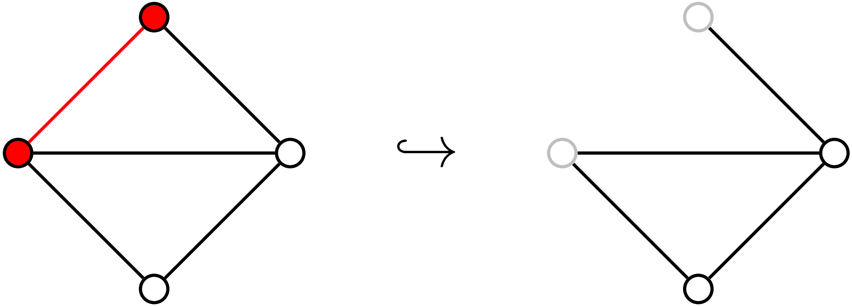

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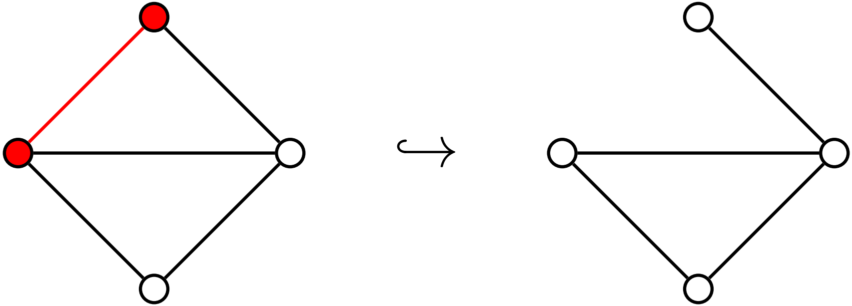

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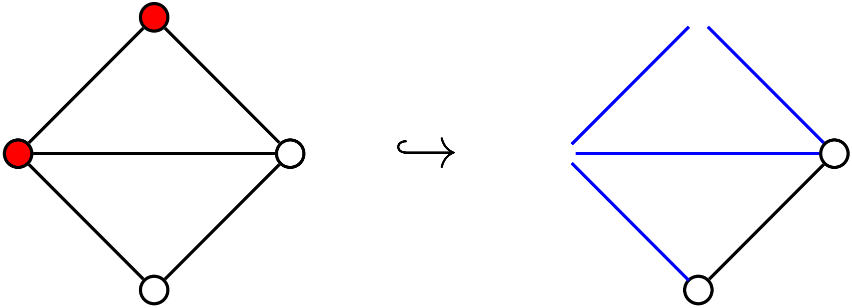

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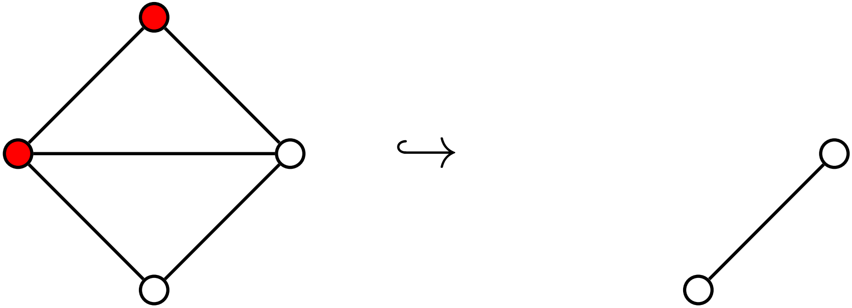

Regular double p-algebras

- Let **A** be an algebra. We say that **A** is *congruence regular* if, for all $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$, we have

$$((\exists x \in A)\ x/\alpha = x/\beta) \implies \alpha = \beta.$$

- Example: groups

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- Example: groups

Theorem (Varlet, 1972)

Let **A** *be a double p-algebra. Then the following are equivalent.*

1. **A** *is congruence regular.*
2. $(\forall a, b \in A)$ *if* $a^* = b^*$ *and* $a^+ = b^+$ *then* $a = b$.
3. $(\forall a, b \in A)\ a \wedge a^+ \leq b \vee b^*$.

A well-behaved structure

Theorem

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Let $G = \langle V, E\rangle$ be a graph. Then $\mathcal{S}(G)$ is (the underlying lattice of) a regular double p-algebra.

Proof.

Let $A = \langle A_V, A_E\rangle$ and $B = \langle B_V, B_E\rangle$ be subgraphs of G. Recall that for a subgraph $H = \langle V', E'\rangle$,

$$H^* = \langle V \backslash V', \{e \in E \backslash E' \mid (\forall x \in e)\; x \in V \backslash V'\}\rangle \tag{1}$$

$$H^+ = \langle V \backslash V' \cup \{v \in V \mid (\exists e \in E \backslash E')\; v \in e\}, E \backslash E'\rangle. \tag{2}$$

Assume $A^* = B^*$ and $A^+ = B^+$. Then from (1) we have $V \backslash A_V = V \backslash B_V$ and from (2) we have $E \backslash A_E = E \backslash B_E$. Hence, $A = B$. □

Some results from the literature

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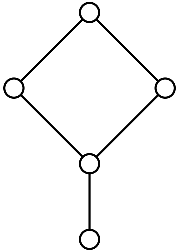
Theorem (Katriňák, 1973)

*Let **A** be a regular double p-algebra. Then **A** is term-equivalent to a double-Heyting algebra via the term*

$$x \to y = (x^* \vee y^{**})^{**} \wedge [(x \vee x^*)^+ \vee x^* \vee y \vee y^*],$$

and its dual.

Are graphs enough?



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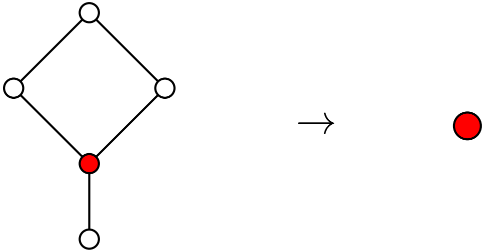

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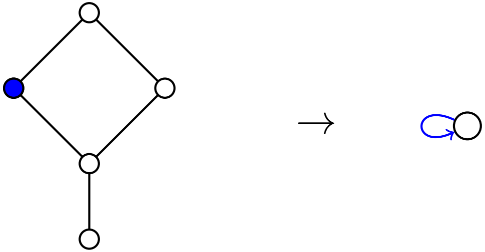

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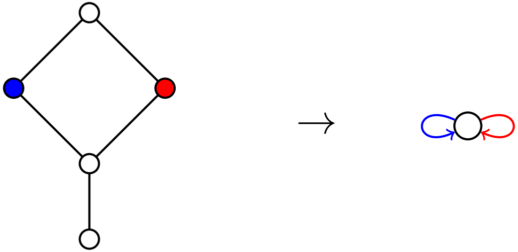

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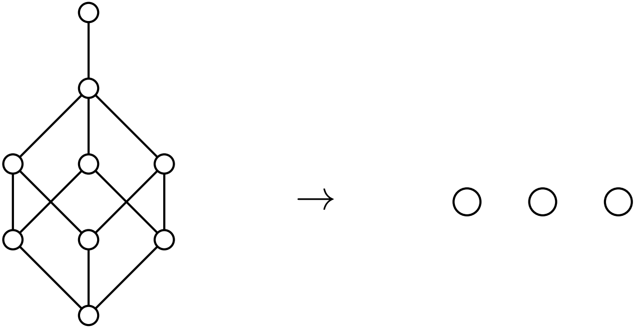

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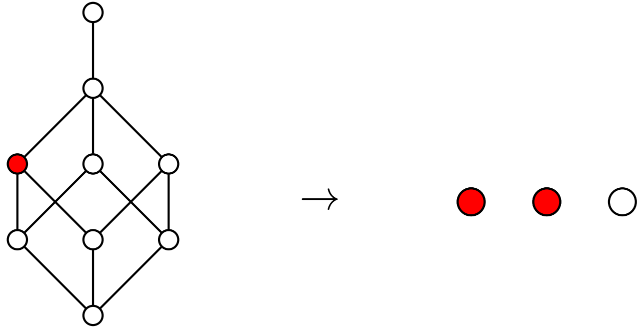

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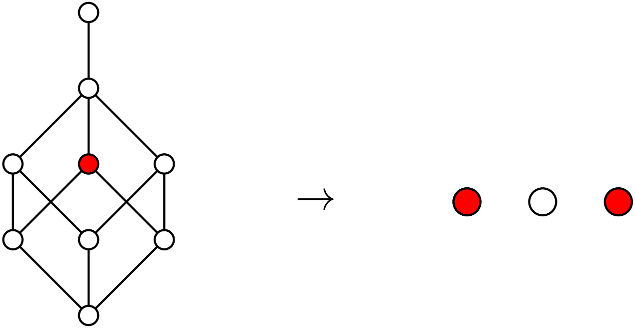

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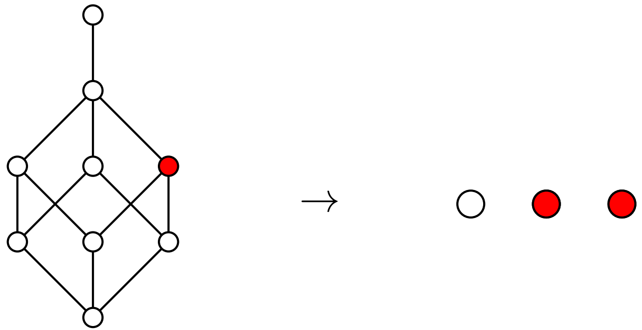

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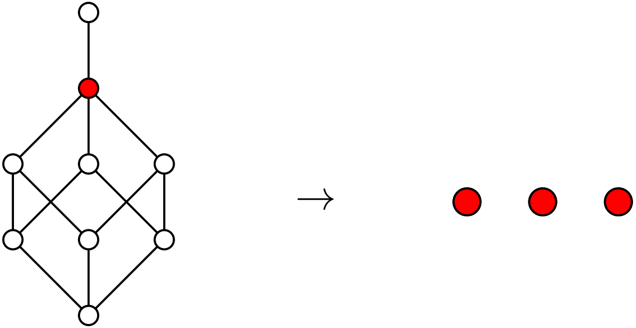

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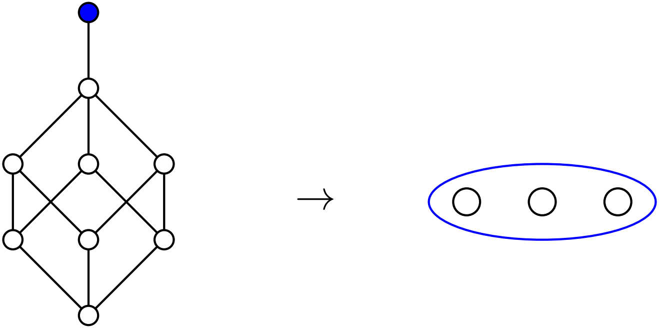

Incidence structures

Definition

An *incidence structure* is a triple $\langle P, L, I\rangle$ where P is a set of points, L is a set of lines and $I \subseteq P \times L$ is an incidence relation describing which points are incident to which lines.

Example

Let $P = \{1, 2, 3\}$, $L = \{x, y, z, a, b\}$, and let

$$\begin{aligned} I = &\{1, 2, 3\} \times \{x, y\} \\ &\cup \{1, 2\} \times \{z\} \\ &\cup \{(1, a), (1, b)\} \end{aligned}$$

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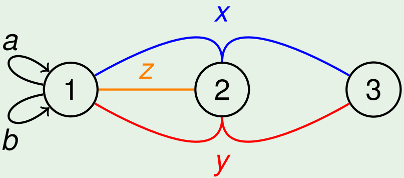

Point-preserving substructures

Definition

Let $G = \langle P, L, I \rangle$ be an incidence structure. A *point-preserving substructure* of G is a pair $\langle P', L' \rangle$ such that

1. $P' \subseteq P$ and $L' \subseteq L$,
2. for all $\ell \in L'$, if $(p, \ell) \in I$ then $p \in P'$.

The incidence relation is defined implicitly from G.

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The incidence relation is defined implicitly from G.

Let $\mathcal{S}(G)$ denote the set of all point-preserving substructures of a structure G. This induces a double p-algebra in a similar way to graphs, where

$$\langle P', L' \rangle^* = \langle P \backslash P', \{\ell \in L \backslash L' \mid (\forall p \in P)\ (p, \ell) \in I \implies p \in P \backslash P'\} \rangle$$
$$\langle P', L' \rangle^+ = \langle P \backslash P' \cup \{p \in P \mid (\exists \ell \in L \backslash L')\ (p, \ell) \in I\}, L \backslash L' \rangle.$$

The main result (finite version)

Theorem

Let L be a finite lattice. Then the following are equivalent.

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Theorem (T., 2015)

Let L be a finite lattice. Then the following are equivalent.

1. *L is (the underlying lattice of) a regular double p-algebra,*
2. *$L \cong \mathcal{S}(G)$ for some incidence structure G,*
3. *$L \cong \mathbf{2}^n \times \mathcal{S}(G)$ for some $n \geq 0$ and some incidence structure G.*

The main result

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Let B be a boolean lattice. Then the following are equivalent.

1. $B \cong \mathcal{P}(X)$ *for some set* X.
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Theorem (T., 2015)

Let $\mathbf{A}$ *be a regular double p-algebra. Then the following are equivalent.*

1. $\mathbf{A} \cong \mathcal{P}(B) \times \mathcal{S}(G)$ *for some set* B *and some incidence structure* G.
2. $\mathbf{A} \cong \mathcal{S}(G)$ *for some incidence structure* G.
3. $\mathbf{A}$ *is complete, completely distributive and doubly atomic.*

Behind the scenes

For a complete lattice **L** we say that **L** *satisfies the join infinite distributivity law* (JID) if the following identity holds:

$$x \wedge \bigvee Y = \bigvee \{x \wedge y \mid y \in Y\}.$$

Dually we say that **A** *satisfies the meet infinite distributivity law* (MID) if the following identity holds:

$$x \vee \bigwedge Y = \bigwedge \{x \vee y \mid y \in Y\}.$$

Theorem (T., 2015)

Let **A** *be a complete doubly atomic regular double p-algebra. If* **A** *satisfies* (JID)*, and for all atoms* $a \in \mathbf{A}$ *and all coatoms* $c \in \mathbf{A}$ *we have* $a \leq c$*, then there is an incidence structure* G *such that* $\mathcal{S}(G) \cong \mathbf{A}$*. Furthermore,* G *has no isolated points and no empty lines.*

The main result

Theorem (T., 2015)

Let **A** *be a regular double p-algebra. The following are equivalent.*

1. **A** $\cong \mathcal{P}(B) \times \mathcal{S}(G)$ *for some set B and some incidence structure G with no isolated points and no empty lines.*
2. **A** $\cong \mathcal{P}(B) \times \mathcal{S}(G)$ *for some set B and some incidence structure G.*
3. **A** $\cong \mathcal{S}(G)$ *for some incidence structure G.*
4. **A** *is complete, completely distributive and doubly atomic.*
5. **A** *is complete, satisfies* (JID) *and* (MID) *and is doubly atomic.*

The main result

Theorem (T., 2015)

Let **A** *be a regular double p-algebra. The following are equivalent.*

1. $\mathbf{A} \cong \mathcal{P}(B) \times \mathcal{S}(G)$ *for some set* B *and some incidence structure* G *with no isolated points and no empty lines.*
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4. **A** *is complete, completely distributive and doubly atomic.*
5. **A** *is complete, satisfies* (JID) *and* (MID) *and is doubly atomic.*

Theorem (T., 2015)

Let **A** *be a regular double p-algebra. Then there is an incidence structure* G *such that* **A** *is isomorphic to a subalgebra of* $\mathcal{S}(G)$*.*