Expansions of dually pseudocomplemented Heyting algebras

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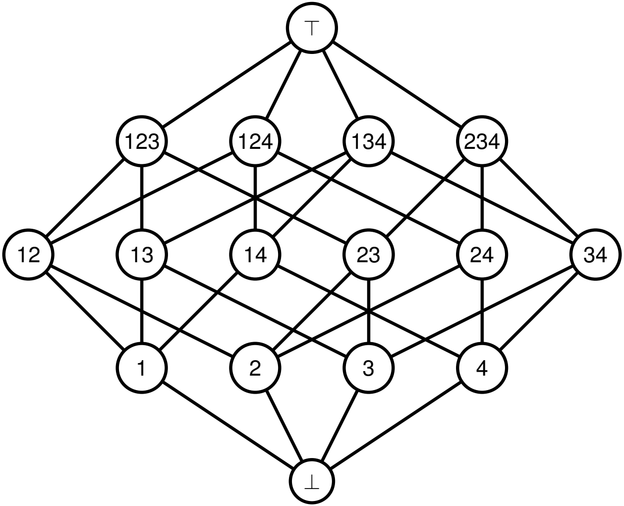
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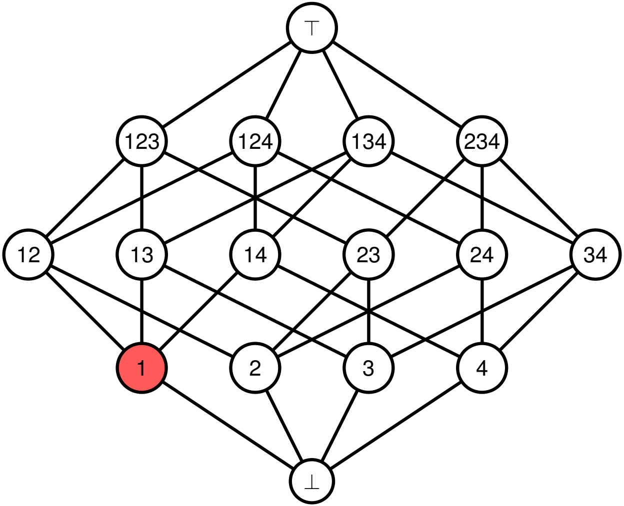
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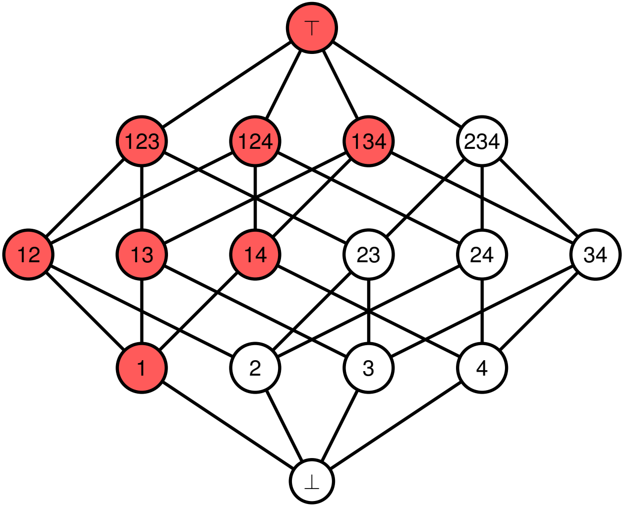
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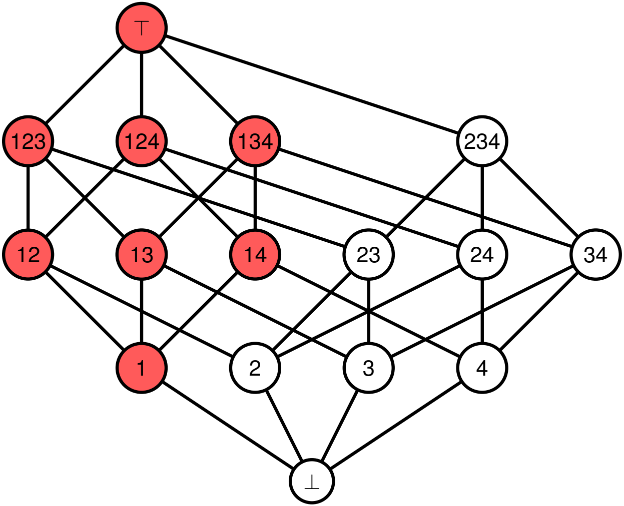
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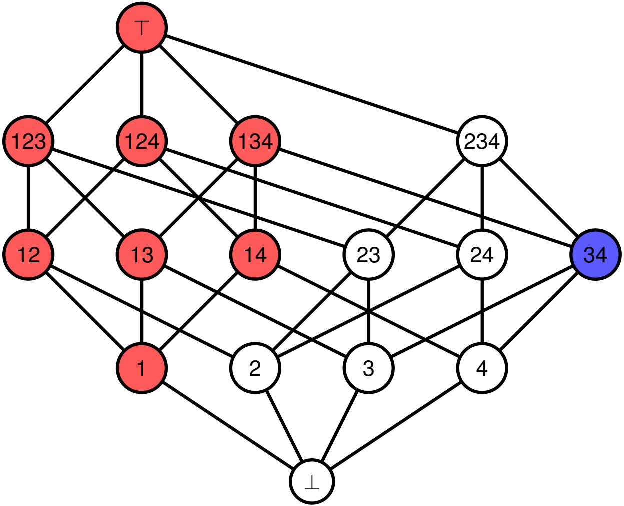
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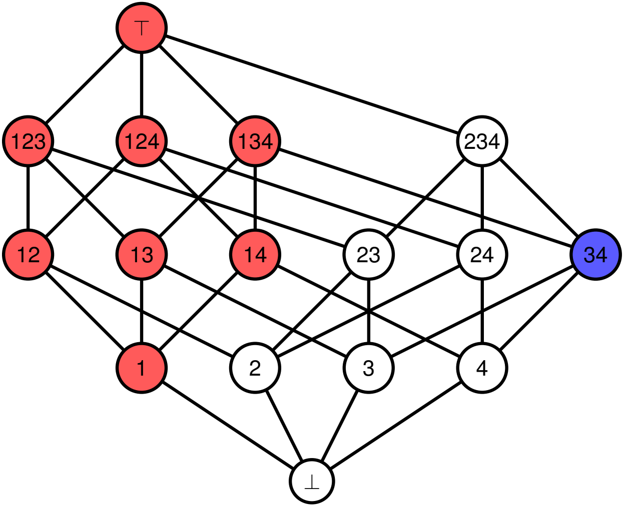
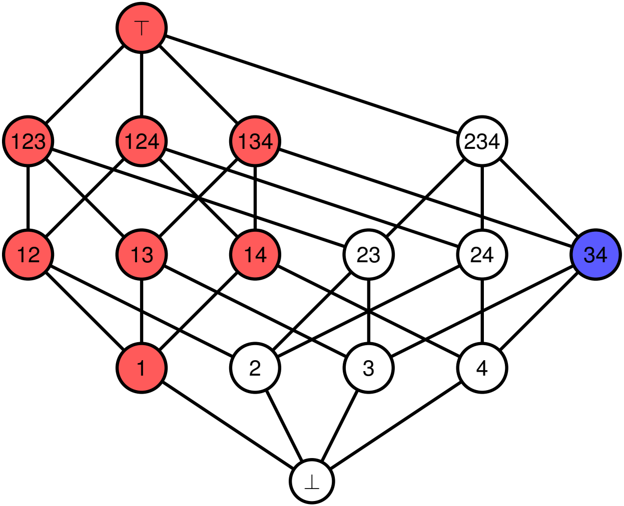


Relative complement of 12 with respect to 1 is equal to 1 ∨ 34.



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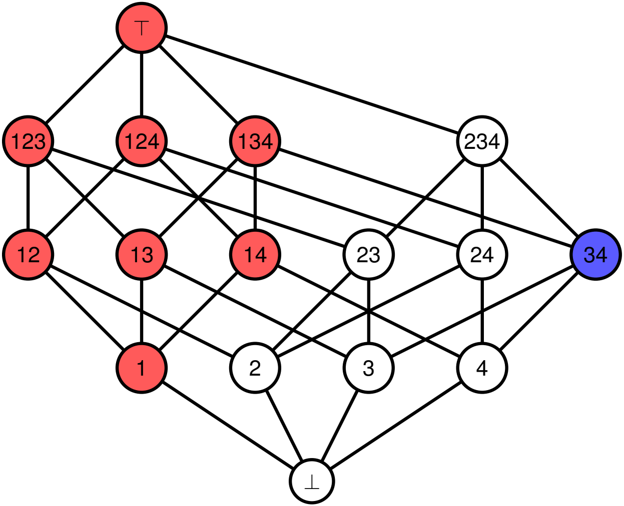

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More generally: relative complement of x with respect to y is $y \vee \neg x$
...also known as $x \rightarrow y$

Heyting algebras

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Theorem

The class of Heyting algebras is an equational class, defined by the equations for bounded distributive lattices, along with:

1. $x \wedge (x \to y) = x \wedge y$,
2. $x \wedge (y \to z) = x \wedge [(x \wedge y) \to (x \wedge z)]$, *and,*
3. $(x \wedge y) \to x = 1$.

Overview

1. Expansions of Heyting algebras
   - Congruences
   - Dually pseudocomplemented Heyting algebras
2. Applications
   - Subdirectly irreducibles
   - Characterising EDPC, semisimplicity and discriminator varieties

Filters

Definition

Let **L** be a lattice and let $F \subseteq L$. Then F is a *filter* provided that:

1. F is an upset, and,
2. if $x, y \in F$ then $x \wedge y \in F$.

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For a Heyting algebra **A**, and a filter F of **A**, the binary relation

$$\theta(F) := \{(x, y) \mid x \leftrightarrow y \in F\}$$

is a congruence on **A**, where $x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$.

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Fundamental theorem of Heyting algebras

Every congruence on a Heyting algebra arises from a filter.

Normal filters

Definition

Let $\mathbf{A}$ be a Heyting algebra, let $f\colon A^n \to A$ be any map and let F be a filter on $\mathbf{A}$. We say that F is *normal with respect to f* if, for all $x_1, y_1, \ldots, x_n, y_n \in A$,

$$\{x_i \leftrightarrow y_i \mid i \leq n\} \subseteq F \implies f(x_1, \ldots, x_n) \leftrightarrow f(y_1, \ldots, y_n) \in F.$$

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Example

If f is unary, then F is normal with respect to f provided that, for all $x, y \in A$, if $x \leftrightarrow y \in F$ then $fx \leftrightarrow fy \in F$.

Expanded Heyting algebras

Definition

An algebra $\mathbf{A} = \langle A; M, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is an *expanded Heyting algebra* (EHA) if the reduct $\langle A; \vee, \wedge, \rightarrow, 0, 1 \rangle$ is a Heyting algebra and M is a set of operations on A.

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Recall that for any filter F on $\mathbf{A}$, the congruence $\theta(F)$ is defined by

$$\theta(F) := \{(x, y) \mid x \leftrightarrow y \in F\},$$

and F is normal with respect to a (unary) map f if,

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Theorem

Let $\mathbf{A}$ be an EHA and let F be a filter on $\mathbf{A}$. Then $\theta(F)$ is a congruence on $\mathbf{A}$ if and only if F is normal with respect to M.

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Throughout the rest of this talk, any unquantified **A** will be a fixed but arbitrary EHA.

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Let t be a unary term in the language of **A**. We say that t is a *normal filter term* (*on* **A**) provided that, for all $x, y \in A$ and every filter F on **A**:

1. if $x \leq y$ then $t^{\mathbf{A}}x \leq t^{\mathbf{A}}y$, and,
2. F is a normal filter if and only if F is closed under $t^{\mathbf{A}}$.

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Example

The identity function is a normal filter term for Heyting algebras.

A richer example – boolean algebras with operators

Definition

Let **A** be a bounded lattice and let f be a unary operation on A. The map f is a (dual normal) *operator* if $f(x \wedge y) = fx \wedge fy$, and, $f1 = 1$.

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An algebra $\mathbf{A} = \langle A; \{f_i \mid i \in I\}, \vee, \wedge, \neg, 0, 1\rangle$ is a *boolean algebra with operators* (BAO) if $\langle A; \vee, \wedge, \neg, 0, 1\rangle$ is a boolean algebra and each f_i is an operator.

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Theorem ("Folklore")

Let $\mathbf{A}$ *be a BAO of finite type. Then the term* t*, defined by*

$$tx = \bigwedge\{f_i x \mid i \in I\}$$

is a normal filter term on $\mathbf{A}$*.*

Constructing normal filter terms

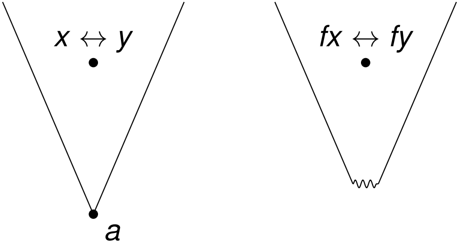
Let **A** be a Heyting algebra and let $f\colon A \to A$ be a unary map. For each $a \in A$, define the set

$$f^{\leftrightarrow}(a) = \{fx \leftrightarrow fy \mid x \leftrightarrow y \geq a\}.$$

Now define the partial operation $[M]$ by

$$[M]a = \bigwedge \bigcup \{f^{\leftrightarrow}(a) \mid f \in M\}.$$

If it is defined everywhere then we say that $[M]$ *exists in* **A**.



Constructing normal filter terms

Recall that M is the set of extra operations on the Heyting algebra.

Lemma (Hasimoto, 2001)

If $[M]$ *exists, then* $[M]$ *is a* (*dual normal*) *operator.*

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Lemma (T., 2016)

If there exists a term t in the language of $\mathbf{A}$ such that $t^{\mathbf{A}}x = [M]x$, then t is a normal filter term.

Constructing normal filter terms

Definition

Let **A** be a Heyting algebra and let f be a unary operation on A. The map f is an *anti-operator* if $f(x \wedge y) = fx \vee fy$, and, $f1 = 0$. Let $\neg x$ be the unary term defined by $\neg x = x \to 0$.

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Lemma (T., 2016)

Let **A** *be an EHA and let f be an anti-operator on A. Then $[f]$ exists, and*

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Example (Meskhi, 1982)

If $\mathbf{A}$ is a Heyting algebra with involution, i.e. a Heyting algebra equipped with a single unary operation i that is a dual automorphism. The map $tx := \neg ix$ is a normal filter term on $\mathbf{A}$.

The dual pseudocomplement

Example

Let **A** be an EHA. A unary operation $\sim$ is a *dual pseudocomplement operation* if the following equivalence is satisfied for all $x \in A$:

$$x \vee y = 1 \iff y \geq {\sim}x.$$

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Corollary (Sankappanavar, 1985)

Let **A** *be a dually pseudocomplemented Heyting algebra. Then* $\neg\sim$ *is a normal filter term on* **A**.

Subdirectly irreducibles

Lemma

Let $\mathbf{A}$ *be an EHA, let* t *be a normal filter term on* $\mathbf{A}$*, and let* $dx = x \wedge tx$*. Then* $(y, 1) \in \mathrm{Cg}^{\mathbf{A}}(x, 1)$ *if and only if* $y \geq d^n x$ *for some* $n \in \omega$*.*

Lemma

Let $\mathbf{A}$ *be an EHA, let* t *be a normal filter term on* $\mathbf{A}$*, and let* $dx = x \wedge tx$*.*

1. $\mathbf{A}$ *is subdirectly irreducible if and only if there exists* $b \in A \backslash \{1\}$ *such that for all* $x \in A \backslash \{1\}$ *there exists* $n \in \omega$ *such that* $d^n x \leq b$*.*
2. $\mathbf{A}$ *is simple if and only if for all* $x \in A \backslash \{1\}$ *there exists* $n \in \omega$ *such that* $d^n x = 0$*.*

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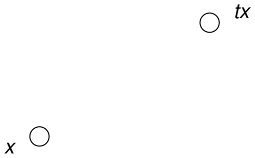

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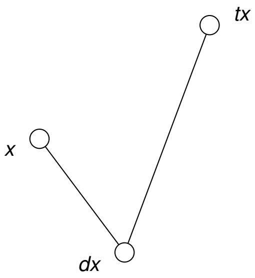

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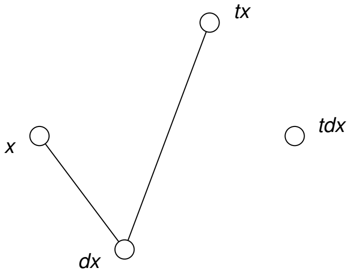

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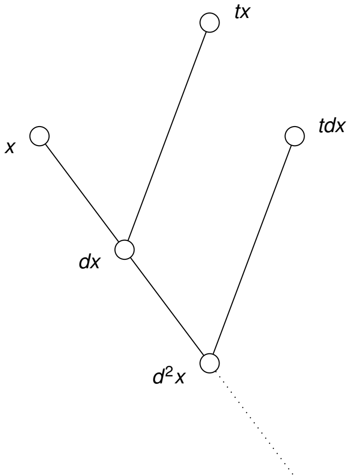

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2. $\mathbf{A}$ *is simple if and only if for all* $x \in A \backslash \{1\}$ *there exists* $n \in \omega$ *such that* $d^n x = 0$*.*

EDPC

Definition

A variety $\mathcal{V}$ has *definable principal congruences* (DPC) if there exists a first-order formula $\varphi(x, y, z, w)$ in the language of $\mathcal{V}$ such that, for all $\mathbf{A} \in \mathcal{V}$, and all $a, b, c, d \in A$, we have

$$(a, b) \in \mathrm{Cg}^{\mathbf{A}}(c, d) \iff \mathbf{A} \models \varphi(a, b, c, d).$$

If φ is a finite conjunction of equations then $\mathcal{V}$ has *equationally definable principal congruences* (EDPC).

Theorem (T., 2016)

Let $\mathcal{V}$ be a variety of EHAs with a common normal filter term t, and let $dx = x \wedge tx$. Then the following are equivalent:

1. *$\mathcal{V}$ has EDPC,*
2. *$\mathcal{V}$ has DPC,*
3. *$\mathcal{V} \models d^{n+1}x = d^n x$ for some $n \in \omega$.*

Discriminator varieties

Definition

A variety is *semisimple* if every subdirectly irreducible member of $\mathcal{V}$ is simple. If there is a ternary term t in the language of $\mathcal{V}$ such that t is a discriminator term on every subdirectly irreducible member of $\mathcal{V}$, i.e.,

$$t(x, y, z) = \begin{cases} x & \text{if } x \neq y \\ z & \text{if } x = y, \end{cases}$$

then $\mathcal{V}$ is a *discriminator variety*.

Theorem (Blok, Köhler and Pigozzi, 1984)

Let $\mathcal{V}$ be a variety of any signature. The following are equivalent:

1. *$\mathcal{V}$ is semisimple, congruence permutable, and has EDPC.*
2. *$\mathcal{V}$ is a discriminator variety.*

The main result

Theorem (T., 2016)

Let $\mathcal{V}$ be a variety of dually pseudocomplemented EHAs, assume $\mathcal{V}$ has a normal filter term t, and let $dx = \neg{\sim}x \wedge tx$. Then the following are equivalent.

1. *$\mathcal{V}$ is semisimple.*
2. *$\mathcal{V}$ is a discriminator variety.*
3. *$\mathcal{V}$ has DPC and there exists $m \in \omega$ such that $\mathcal{V} \models x \leq d{\sim}d^m\neg x$.*
4. *$\mathcal{V}$ has EDPC and there exists $m \in \omega$ such that $\mathcal{V} \models x \leq d{\sim}d^m\neg x$.*
5. *There exists $n \in \omega$ such that $\mathcal{V} \models d^{n+1}x = d^n x$ and $\mathcal{V} \models d{\sim}d^n x = {\sim}d^n x$.*

This generalises a result by Kowalski and Kracht (2006) for BAOs and a result by the author to appear for double-Heyting algebras.