Double Heyting Algebras

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Contents

Summary

A double Heyting algebra is an algebra comprising both a Heyting algebra and a dual Heyting algebra. This thesis is a general study of properties of double Heyting algebras. We begin with some examples of double Heyting algebras, and we see that they have some natural connections with graph theory. For instance, the homomorphism lattice of finite graphs forms a double Heyting algebra, and the lattice of subgraphs of a graph forms a double Heyting algebra. After we characterise subgraph lattices, we look at congruences of double Heyting algebras. It turns out that they have more in common with Boolean algebras with operators than with Heyting algebras. We formalise that connection by defining a framework that ties the two algebraic structures together. Specifically, we investigate algebras with a Heyting algebra reduct that have their congruences determined by a single unary term.

The framework just described is the basis for many of the main results in this thesis. Accordingly, we prove some sufficient conditions for a class of algebras to be endowed with such a term. Despite the simplicity of the setup, it yields powerful tools. We characterise subdirectly irreducible algebras and varieties with equationally definable principal congruences. We also provide a technique that can be used to prove that a finite algebra is not a splitting algebra. In the presence of a dual pseudocomplement operation, stronger results are obtained. Specifically, we prove that a variety of such algebras is a discriminator variety if and only if it is semisimple. All of these results apply to double Heyting algebras and finite-signature Boolean algebras with operators. We also apply the results to some algebras that are neither double Heyting algebras nor Boolean algebras with operators, obtaining some old results from the literature as corollaries, as well as some new results. Towards the end of the thesis, we return the focus to the variety of double Heyting algebras and prove various results about its lattice of subvarieties. This includes a proof that there are exactly two splitting double Heyting algebras. The overall structure of the lattice of subvarieties of double Heyting algebras is still largely unknown.

Statement of Authorship

Except where reference is made in the text of the thesis, this thesis contains no other material published elsewhere or extracted in whole or in part from a thesis accepted for the award of any other degree or diploma. No other person's work has been used without due acknowledgment in the main text of the thesis. This thesis has not been submitted for the award of any degree or diploma in any other tertiary institution.

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Introduction

Heyting algebras were introduced as a formalisation of intutionistic logic in 1930 by Arend Heyting [44–46]. Since then, Heyting algebras have been subject to intense investigation. The lattice-theoretic definition lends itself to being dualised, and an algebra comprising both a Heyting algebra and a dual Heyting algebra is called a *double Heyting algebra*. The earliest reference to double Heyting algebras we have found was in 1971 by Rauszer [77], where they were called semi-Boolean algebras. They have also been called bi-Heyting algebras [36, 41, 78] and Heyting–Brouwer algebras [49, 93]. Compared to Heyting algebras, double Heyting algebras have spawned lesser interest. This is perhaps because, while the success of Heyting algebras lies in their connection with logic, a logic algebraised by double Heyting algebras is comparatively artificial. However, bearing in mind the duality principle for lattices, double Heyting algebras are a perfectly natural structure from an algebraic point of view. Our investigation began as such: as mathematics that is interesting in its own right. But in the course of writing this thesis we have uncovered some connections between double Heyting algebras and graph theory. These connections lead into some interesting results and currently unexplored pathways, and we retroactively cite this as our primary motivation.

Central to this thesis is an understanding of the congruences on a double Heyting algebra. Congruences on a Heyting algebra are in one-to-one correspondence with filters of the underlying lattice, and this implies that double Heyting algebra congruences are determined by some kind of filter. They were characterised by Köhler [59] as filters closed under a particular unary term in the language of double Heyting algebras. Beazer [4] further characterised subdirectly irreducible and simple double Heyting algebras in terms of that term. A close relative of double Heyting algebras, which we will call a *dually pseudocomplemented Heyting algebra* (or H^+-algebra for short), was first introduced by Sankappanavar [79]. Every double Heyting algebra has an underlying H^+-algebra, and the congruences of the latter structure are identical to the congruences of the former. This actually places the structures into an even more general framework: algebras with a Heyting algebra reduct with their congruences determined by a unary term. This framework underlies a large portion of this thesis and assimilates with the theory of finite-signature Boolean algebras with operators.

Throughout this thesis we will assume familiarity with standard universal algebraic concepts and Priestley's duality for distributive lattices. In Chapter 1, we recall the necessary algebraic background, cement our notation, and state equational definitions for the main characters in this thesis.

In Chapter 2, we cover Priestley's duality for distributive lattices as well as its restrictions to Heyting algebras and double Heyting algebras. We also consider some properties of morphisms on finite ordered sets.

In Chapter 3, we look at some general properties of double Heyting algebras. This includes sufficient conditions ensuring a lattice forms a double Heyting algebra. We briefly look at splittings of double Heyting algebras. The rest of the chapter explores some concrete examples of double Heyting algebras:

- Using an argument by Brian Davey, we prove that the lattice of subvarieties of a locally finite congruence-distributive variety forms a double Heyting algebra.
- The lattice of open sets of a topological space always forms a Heyting algebra, so we consider conditions for it to also form a double Heyting algebra.
- The lattice of subgraphs of a graph is easily shown to be a double Heyting algebra, where pseudocomplements and dual pseudocomplements have a very natural interpretation as two types of graph complements.
- In the last section of the chapter, we use an argument by Brian Davey to prove that a well known Heyting algebra known as the *digraph homomorphism lattice* is in fact a double Heyting algebra.

In Chapter 4, we return to lattices of subgraphs. We prove that the lattice of subgraphs of a graph actually forms a (congruence-)regular double p-algebra, which is term-equivalent to a double Heyting algebra by a result of Katriňák [55]. An obvious question now is, which regular double p-algebras are isomorphic to a subgraph lattice? This is best approached more generally by considering incidence structures, which are a standard generalisation of multigraphs, hypergraphs, and projective planes. With substructures defined appropriately, we verify that the set of substructures of an incidence structure is a regular double p-algebra, and we characterise them as completely distributive and doubly atomic regular double p-algebras. In particular, every finite regular double p-algebra is isomorphic to the substructure lattice of an incidence structure. The result is similar to the case for Boolean algebras, wherein powerset lattices are characterised as completely distributive atomic Boolean algebras.

In Chapter 5, we investigate expansions of Heyting algebras, by which we mean algebras with a Heyting algebra reduct. We bind together double Heyting algebras and finite-signature Boolean algebras with operators by introducing *congruence-filter terms*. A congruence-filter term, when it exists, is a unary term that determines the filters that determine congruences of an algebra with a Heyting algebra reduct. We prove some sufficient conditions guaranteeing that certain classes of algebras have a congruence-filter term. In the presence of a congruence-filter term, it is easy to characterise subdirectly irreducible algebras. The theory substantially generalises results for double Heyting algebras and H^+-algebras by Köhler [59], Beazer [4], and Sankappanavar [79]. In the last two sections of Chapter 5 we look at varieties of algebras with a congruence-filter term. We characterise the subvarieties that have equationally definable principle congruences and provide a technique to prove that a finite algebra is not a splitting algebra.

In Chapter 6, we expand on the previous chapter by assuming further that the signature includes a dual pseudocomplement operation. This includes double Heyting algebras and H^+-algebras. As far as congruences are concerned, for Heyting algebras, filters cannot be interchanged with ideals. But the symmetry of double Heyting algebras means that the choice between filters and ideals is completely arbitrary. This lets us prove that a larger class of algebras have a congruence-filter term, provided that the algebra also has a double Heyting algebra reduct. Despite the lack of symmetry in the operations, we will prove that the same symmetry between filters and ideals exists for congruences on expansions of H^+-algebras. Unfortunately, it does not enable us to extend the existence conditions for congruence-filter terms. The rest of the chapter is a proof that if a variety of dually pseudocomplemented algebras has a congruence-filter term, then it is semisimple if and only if it is a discriminator variety.

In Chapter 7, we present some examples of algebras with congruence-filter terms, and we apply the results of Chapter 5 and Chapter 6. For double Heyting algebras, the equivalence of semisimple varieties and discriminator varieties extends to varieties with equationally definable principal congruences as well. We prove the same thing for De Morgan–Heyting algebras. We also consider Heyting algebras with operators, Boolean algebras with operators, and symmetric Heyting relation algebras, and we prove that various results from the literature are corollaries of the general theory in Chapter 5 and Chapter 6.

In Chapter 8, we return the focus to pure H^+-algebras and double Heyting algebras. In particular, we look at the lattice of subvarieties of the two varieties. It is easily shown that in both cases the trivial subvariety has a unique cover, namely the variety of Boolean algebras. We prove that the variety of Boolean algebras has a unique cover as well, which is generated by the three-element chain. Covers of that subvariety have not yielded to our investigation. We also prove, using the method of Chapter 5, that there are exactly two splitting algebras in the variety of double Heyting algebras and the variety of H^+-algebras. In the last section of the chapter, by applying some results of Ball and Pultr [3], we showcase an infinite number of subvarieties that contain exactly two finite splitting algebras.

In Chapter 9, we explore some unresolved questions, offering some reformulations and potential strategies. We compare the lattice of subvarieties of H^+-algebras to the lattice of subvarieties of double Heyting algebras. Then we translate a long-standing open problem in graph theory into a new question involving lattices. We finish by briefly revisiting the subvariety lattices of H^+-algebras, double Heyting algebras, and regular double p-algebras.

This thesis has various ehancements for PDF display. Inline references are hyperlinked, and clicking on them will bring the reader to the appropriate page. This includes citations, which are linked to the bibliography. Moreover, when available, each reference in the bibliography has a link to its MathSciNet review and a DOI reference.

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Heyting algebras and their cousins

This chapter's primary purpose is to lay out our notation and introduce the central objects of this thesis. We will assume familiarity with lattice theory and treat it as assumed knowledge. We will also assume familiarity with standard results of universal algebra. Our main references for lattice theory are Davey and Priestley [26] and Balbes and Dwinger [2]. The main reference for universal algebra is Burris and Sankappanavar [17]. Our notation will follow that of [17] and [26].

In the first section of this chapter we will briefly describe our notation and remark on other conventions. The rest of the chapter is an outline of double Heyting algebras, dually pseudocomplemented Heyting algebras, Heyting algebras, and p-algebras. We include equational characterisations and some useful properties that we will use frequently and without reference. The properties of pseudocomplements and dual pseudocomplements are crucial throughout this thesis. The reader may refer to Balbes and Dwinger [2] for more on Heyting algebras and refer to Blyth [13] for more on p-algebras. The results in this chapter are stated without proof, and there is no original work. Proofs can be found in the associated references. We assume the axiom of choice.

1.1 Algebraic preliminaries

Definition 1.1.1. Let $\mathbf{L}$ be a lattice and let $x, y \in L$. If $\mathbf{L}$ is bounded, then 0 denotes the bottom element and 1 denotes the top. If y covers x, then we write $x \prec y$ and say that x is a *lower cover* of y. If arbitrary joins and meets exist in $\mathbf{L}$, then it is *complete*. If $\mathbf{L}$ is complete, then x is *compact* provided that, for every subset $S \subseteq L$, if $x \leq \bigvee S$, then there is a finite subset T of S such that $x \leq \bigvee T$. Moreover, $\mathbf{L}$ is *algebraic* if it is complete and every element in $\mathbf{L}$ is the join of all compact elements below it. If $\mathbf{L}$ is complete, then $\mathbf{L}$ is *completely distributive* if, for every doubly indexed set $\{x_{i,j} \mid i \in I, j \in J\}$,

$$\bigwedge_{i\in I}\bigvee_{j\in J} x_{i,j} = \bigvee_{f\in F}\bigwedge_{i\in I} x_{i,f(i)},$$

where F is the set of all functions from I to J. We also require two special cases of complete distributivity, namely the *join-infinite distributive law* (JID) and the *meet-infinite distributive law* (MID):

$$x \wedge \bigvee Y = \bigvee\{x \wedge y \mid y \in Y\}, \tag{JID}$$

$$x \vee \bigwedge Y = \bigwedge\{x \vee y \mid y \in Y\}. \tag{MID}$$

We say that $\mathbf{L}$ *satisfies* (JID) or (MID) if it is complete and satisfies the appropriate law above, for all $\{x\} \cup Y \subseteq L$. If $\mathbf{L}$ is bounded, then the center of $\mathbf{L}$ is denoted by $\mathrm{Cen}(\mathbf{L})$. If $\mathbf{L}$ is a bounded distributive lattice, then $\mathrm{Cen}(\mathbf{A})$ is the set of complemented elements of $\mathbf{L}$.

Our algebraic notation is outlined in the next definition.

Definition 1.1.2. Let $\mathbf{A}$ be an algebra and let $x, y \in A$. The set of congruences is denoted by $\mathrm{Con}(\mathbf{A})$, and the lattice of congruences is denoted by $\mathbf{Con}(\mathbf{A})$. The principal congruence generated by x and y is denoted by $\mathrm{Cg}^{\mathbf{A}}(x, y)$. For a congruence α, the block containing x will be denoted by x/α. If $\mathbf{Con}(\mathbf{A})$ is distributive, then $\mathbf{A}$ is *congruence-distributive.* If $\alpha \circ \beta = \beta \circ \alpha$, for all $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$, then $\mathbf{A}$ is *congruence-permutable.* A class $\mathcal{K}$ of algebras is *congruence-distributive* if every algebra in $\mathcal{K}$ is congruence-distributive, and a *congruence-permutable* class is defined analogously. We say that $\mathbf{A}$ has the *congruence extension property* if, for every subalgebra $\mathbf{B}$ of $\mathbf{A}$ and every $\alpha \in \mathrm{Con}(\mathbf{B})$, there exists $\beta \in \mathrm{Con}(\mathbf{A})$ such that $\alpha = \beta \cap B^2$. A class $\mathcal{K}$ of algebras has the *congruence extension property* if every algebra in $\mathcal{K}$ has the congruence extension property. We denote the trivial congruence (that is, the identity relation) by $0_{\mathbf{A}}$ and the full congruence by $1_{\mathbf{A}}$. The algebra $\mathbf{A}$ is *subdirectly irreducible* if there is a minimum element of $\mathrm{Con}(\mathbf{A})\backslash\{0_{\mathbf{A}}, 1_{\mathbf{A}}\}$, in which case that element is called the *monolith*, and $\mathbf{A}$ is *simple* if $\mathrm{Con}(\mathbf{A})$ has exactly two elements.

Notice that our definition of subdirectly irreducible and simple algebras excludes one-element algebras. It is a standard result that congruence lattices are algebraic, wherein the finitely generated congruences are compact. This includes principal congruences as a special case. Moreover, if $\mathbf{A}$ is congruence-permutable, then $\alpha \vee \beta = \alpha \circ \beta$, for all $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$.

Definition 1.1.3. Let $\mathcal{K}$ be a class of similar algebras. We let $\mathsf{I}(\mathcal{K})$, $\mathsf{H}(\mathcal{K})$, $\mathsf{S}(\mathcal{K})$, $\mathsf{P}(\mathcal{K})$, and $\mathsf{P_U}(\mathcal{K})$ denote respectively the classes of isomorphic copies, homomorphic images, subalgebras, products, and ultraproducts of algebras in $\mathcal{K}$. For two algebras $\mathbf{A}$ and $\mathbf{B}$, we write $\mathbf{A} \leq \mathbf{B}$ if $\mathbf{A} \in \mathsf{IS}(\mathbf{B})$. The class of subdirectly irreducible algebras in $\mathcal{K}$ is denoted by $\mathrm{Si}(\mathcal{K})$. The variety generated by $\mathcal{K}$ is denoted by $\mathrm{Var}(\mathcal{K})$.

Throughout this thesis, we will refer to the following important theorem as *Jónsson's Lemma*.

Theorem 1.1.4 (Jónsson [52]). *Let $\mathcal{V}$ be a congruence-distributive variety.*

(1) *If $\mathcal{K} \subseteq \mathcal{V}$, then $\mathrm{Si}(\mathrm{Var}(\mathcal{K})) \subseteq \mathsf{HSP_U}(\mathcal{K})$.*

(2) *If $\mathcal{K} \subseteq \mathcal{V}$ and $\mathcal{K}$ is a finite set of finite algebras, then $\mathrm{Si}(\mathrm{Var}(\mathcal{K})) \subseteq \mathsf{HS}(\mathcal{K})$, so there are finitely many subdirectly irreducible algebras in $\mathrm{Var}(\mathcal{K})$.*

(3) *If $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{V}$, then $\mathrm{Si}(\mathrm{Var}(\mathcal{K}_1 \cup \mathcal{K}_2)) = \mathrm{Si}(\mathrm{Var}(\mathcal{K}_1)) \cup \mathrm{Si}(\mathrm{Var}(\mathcal{K}_2))$.*

Remark 1.1.5. Some other conventions we will follow are listed below.

(1) Unary operations bind stronger than anything. For higher arity operations, bracketing will be used.

(2) The symbol $\mathbb{N}$ denotes the set of natural numbers not including 0, and ω is the set of natural numbers including 0.

(3) If t is a symbol interpreted as a unary term or a unary function, then, for each $n \in \omega$, the n-th iteration of t is denoted by $t^n x$. More formally, let $t^0 x = x$ and, for each $n \in \omega$, let $t^{n+1} x = t(t^n x)$.

(4) The two-element chain is denoted by $\mathbf{2}$, and the three-element chain is denoted by $\mathbf{3}$. Any algebraic structure on the chains will be determined by the context.

1.2 Heyting algebras

Definition 1.2.1. Let $\mathbf{L}$ be a lattice and let $x, y \in L$. The *relative pseudocomplement of x with respect to y* is an element $x \to y \in L$ satisfying

$$x \wedge y \leq z \iff x \leq y \to z.$$

If $x \to y$ exists, for all $x, y \in L$, then $\mathbf{L}$ is a *relatively pseudocomplemented lattice*. An algebra $\langle A; \vee, \wedge, \to, 0, 1\rangle$ is a *Heyting algebra* if the reduct $\langle A; \vee, \wedge, 0, 1\rangle$ is a bounded relatively pseudocomplemented lattice and $\to$ is the relative pseudocomplement operation.

Theorem 1.2.2. *The class of Heyting algebras is an equational class defined by*

(1) *a set of identities defining bounded distributive lattices,*

(2) $x \wedge (x \to y) = x \wedge y$,

(3) $x \wedge (y \to z) = x \wedge [(x \wedge y) \to (x \wedge z)]$,

(4) $x \wedge [(y \wedge z) \to y] = x$.

Lemma 1.2.3. *Let* $\mathbf{L}$ *be a lattice and let* $x, y, z \in L$. *Whenever the elements exist in* $\mathbf{L}$, *the following properties hold.*

(1) $x \leq y$ *if and only if* $x \to y = 1$.

(2) $y \leq x \to y$.

(3) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

(4) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.

(5) *If* $x \leq y$, *then* $z \to x \leq z \to y$ *and* $x \to z \geq y \to z$.

Definition 1.2.4. Let $\mathbf{L}$ be a lattice and let F be a filter of $\mathbf{L}$. Define the binary relation $\theta(F)$ by

$$\theta(F) = \{(x, y) \in A^2 \mid (\exists z \in F)\ x \wedge z = y \wedge z\}.$$

It is well known that a lattice $\mathbf{L}$ is distributive if and only if, for every filter F of $\mathbf{L}$, the relation $\theta(F)$ is a lattice congruence (see [26, Exercise 6.4]). For a Heyting algebra, these are the only congruences.

Theorem 1.2.5. *Let* $\mathbf{A}$ *be a Heyting algebra. Then* $\mathbf{Con}(\mathbf{A})$ *is isomorphic to the lattice of filters of* $\mathbf{A}$. *The isomorphism is given by the map* $F \mapsto \theta(F)$, *and its inverse is given by* $\alpha \mapsto 1/\alpha$.

Definition 1.2.6. Let $\mathbf{A}$ be a Heyting algebra and let $x, y \in A$. We let $x \leftrightarrow y$ be an abbreviation for $(x \to y) \wedge (y \to x)$.

Lemma 1.2.7. *Let* $\mathbf{A}$ *be a Heyting algebra and let* $x, y \in A$. *Then* $x \leftrightarrow y$ *is the largest* $z \in A$ *such that* $x \wedge z = y \wedge z$. *That is,*

$$x \wedge z = y \wedge z \iff z \leq x \leftrightarrow y.$$

Moreover, $x \leftrightarrow y = 1$ *if and only if* $x = y$.

For Heyting algebras, $\theta(F)$ is frequently expressed in the following form.

Lemma 1.2.8. *If* F *is a filter of a Heyting algebra* $\mathbf{A}$, *then*

$$\theta(F) = \{(x, y) \in A^2 \mid x \leftrightarrow y \in F\}.$$

1.3 Pseudocomplements and dual pseudocomplements

Definition 1.3.1. Let $\mathbf{L}$ be a bounded lattice and let $x \in L$. The *pseudocomplement* of x in $\mathbf{L}$ is an element $\neg x \in L$ satisfying

$$x \wedge y = 0 \iff y \leq \neg x.$$

Thus, $\neg x = x \to 0$. Similarly, the *dual pseudocomplement* of x is an element ${\sim}x \in L$ satisfying

$$x \vee y = 1 \iff y \geq {\sim}x.$$

If $\neg x$ exists, for all $x \in L$, then $\mathbf{L}$ is a *pseudocomplemented lattice*. It is a *dually pseudocomplemented lattice* if ${\sim}x$ exists, for all $x \in L$. A *p-algebra* is an algebra $\langle A; \vee, \wedge, \neg, 0, 1\rangle$ such that $\langle A; \vee, \wedge, 0, 1\rangle$ is a pseudocomplemented lattice and $\neg$ is the pseudocomplement operation. A *dual p-algebra* is defined dually. An algebra $\langle A; \vee, \wedge, \neg, {\sim}, 0, 1\rangle$ is a *double p-algebra* if $\langle A; \vee, \wedge, \neg, 0, 1\rangle$ is a p-algebra and $\langle A; \vee, \wedge, {\sim}, 0, 1\rangle$ is a dual p-algebra.

Remark 1.3.2. In the literature it is also common to see the notation x^* for $\neg x$ and x^+ for ${\sim}x$. We have opted for the prefix notation and find that it vastly improves readability, significantly so in Section 6.2.

Theorem 1.3.3. *The class of all p-algebras is an equational class defined by*

(1) *a set of identities defining bounded lattices,*

(2) $x \wedge \neg(x \wedge y) = x \wedge \neg y$,

(3) $\neg 0 = 1$,

(4) $\neg\neg 0 = 0$.

Neither p-algebras nor double p-algebras need to be distributive. By combining the identities above with their duals, the class of double p-algebras is also an equational class.

Lemma 1.3.4. *Let $\mathbf{L}$ be a bounded lattice and let $x, y \in L$. Whenever the elements exist in $\mathbf{L}$, the following properties and their duals hold.*

(1) *If $x \leq y$, then $\neg x \geq \neg y$ and $\neg{\sim}x \leq \neg{\sim}y$.*

(2) $x \leq \neg\neg x$.

(3) $\neg x = \neg\neg\neg x$.

(4) $\neg(x \vee y) = \neg x \wedge \neg y$.

(5) $\neg x = 1$ *if and only if* $x = 0$.

(6) $\neg{\sim}x = 1$ *if and only if* $x = 1$.

(7) $\neg{\sim}(x \wedge y) = \neg{\sim}x \wedge \neg{\sim}y$.

(8) *If* $\mathbf{L}$ *is distributive, then*

  (i) $\neg{\sim}x \leq {\sim}{\sim}x \leq x \leq \neg\neg x \leq {\sim}\neg x$,

  (ii) $\neg x \leq {\sim}x$,

  (iii) $\neg x = {\sim}x$ *if and only if* $\neg{\sim}x = x$ *if and only if* $x \in \mathrm{Cen}(\mathbf{A})$.

1.4 Dually pseudocomplemented Heyting algebras

Definition 1.4.1. An algebra $\langle A; \vee, \wedge, \rightarrow, {\sim}, 0, 1\rangle$ is a *dually pseudocomplemented Heyting algebra* (H$^+$-algebra for short) if $\langle A; \vee, \wedge, \rightarrow, 0, 1\rangle$ is a Heyting algebra and $\langle A; \vee, \wedge, {\sim}, 0, 1\rangle$ is a dual p-algebra.

The name H$^+$-algebra comes from the alternative notation for the dual pseudocomplement. Dually pseudocomplemented Heyting algebras were introduced by Sankappanavar in [79] and are closely connected to double Heyting algebras. We will see in the next section that double Heyting algebras define an H$^+$-algebra reduct and that this reduct determines their congruences.

Theorem 1.4.2. *The class of all H$^+$-algebras is an equational class defined by*

(1) *a set of identities defining bounded distributive lattices,*

(2) $x \wedge (x \rightarrow y) = x \wedge y$,

(3) $x \wedge (y \rightarrow z) = x \wedge [(x \wedge y) \rightarrow (x \wedge z)]$,

(4) $x \wedge [(y \wedge z) \rightarrow y] = x$,

(5) $x \vee {\sim}(x \vee y) = x \vee {\sim}y$,

(6) ${\sim}1 = 0$,

(7) ${\sim}{\sim}1 = 1$.

Definition 1.4.3. Let $\mathcal{H}^+$ denote the variety of dually pseudocomplemented Heyting algebras.

Since they have a Heyting algebra reduct, congruences on an H$^+$-algebra are determined by some kind of filter. These filters were characterised by Sankapannavar. The next theorem will be extensively generalised in Chapter 5, and the resulting generalisation is the foundation for many results in this thesis.

Theorem 1.4.4 (Sankappanavar [79])**.** *Let* $\mathbf{A}$ *be an* H^+*-algebra and let* F *be a filter of* $\mathbf{A}$*. Then* $\theta(F)$ *is a congruence if and only if* F *is closed under* $\neg{\sim}$*.*

Definition 1.4.5. Let $\mathbf{A}$ be an H$^+$-algebra. Then $\mathbf{A}$ is *of finite range* if, for every $x \in A$, there exists $n \in \omega$ such that $(\neg{\sim})^{n+1}x = (\neg{\sim})^n x$. In particular, finite H$^+$-algebras are of finite range.

Corollary 1.4.6 (Sankappanavar [79])**.** *Let* $\mathbf{A}$ *be an* H^+*-algebra of finite range. The following are equivalent:*

(1) $\mathbf{A}$ *is simple;*

(2) $\mathbf{A}$ *is subdirectly irreducible;*

(3) $\mathbf{A}$ *is directly indecomposable;*

(4) $\mathrm{Cen}(\mathbf{A}) = \{0, 1\}$.

Corollary 1.4.7 (Sankappanavar [79])**.** *Let* $\mathbf{A}$ *be an* H^+*-algebra.*

(1) $\mathbf{A}$ *is subdirectly irreducible if and only if there exists* $b \in A\backslash\{1\}$ *such that, for every* $a \in A\backslash\{1\}$*, there exists* $n \in \omega$ *such that* $(\neg{\sim})^n a \leq b$*.*

(2) $\mathbf{A}$ *is simple if and only if, for every* $a \in A\backslash\{1\}$*, there exists* $n \in \omega$ *such that* $(\neg{\sim})^n a = 0$*.*

In both cases, $\mathrm{Cen}(\mathbf{A}) = \{0, 1\}$*.*

Corollary 1.4.8 (Sankappanavar [79])**.** $\mathcal{H}^+$ *has the congruence extension property.*

Corollary 1.4.9 (Sankappanavar [79])**.** *For each* $n \in \omega$*, the subvariety of* $\mathcal{H}^+$ *defined by the identity* $(\neg{\sim})^{n+1}x = (\neg{\sim})^n x$ *is a discriminator variety.*

We reserve the details of discriminator varieties for Chapter 6.

1.5 Double Heyting algebras

Definition 1.5.1. Let $\mathbf{L}$ be a lattice and let $x, y \in L$. The *dual relative pseudo-complement of* x *with respect to* y is an element $y \dot{-} x \in L$ satisfying

$$x \vee z \geq y \iff z \geq y \dot{-} x.$$

In particular, if $\mathbf{L}$ is bounded, then $1 \dot{-} x = {\sim}x$. If $y \dot{-} x$ exists, for all $x, y \in L$, then $\mathbf{L}$ is a *dual relatively pseudocomplemented lattice.* An algebra $\langle A; \vee, \wedge, \dot{-}, 0, 1\rangle$ is a *dual Heyting algebra* if the reduct $\langle A; \vee, \wedge, 0, 1\rangle$ is a bounded dual relatively pseudocomplemented lattice and $\dot{-}$ is the dual relative pseudocomplement operation. A *double Heyting algebra* is an algebra $\langle A; \vee, \wedge, \to, \dot{-}, 0, 1\rangle$ such that $\langle A; \vee, \wedge, \to, 0, 1\rangle$ is a Heyting algebra and $\langle A; \vee, \wedge, \dot{-}, 0, 1\rangle$ is a dual Heyting algebra.

Theorem 1.5.2. *The class of all double Heyting algebras is an equational class defined by*

(1) *a set of identities defining Heyting algebras,*

(2) $x \vee (y \dot{-} x) = x \vee y$,

(3) $x \vee (y \dot{-} z) = x \vee [(x \vee y) \dot{-} (x \vee z)]$,

(4) $z \vee [x \dot{-} (x \vee y)] = z$.

Definition 1.5.3. Let $\mathcal{DH}$ denote the variety of double Heyting algebras.

For absolute clarity, we write down explicitly the dual of Lemma 1.2.3.

Lemma 1.5.4. *Let* $\mathbf{L}$ *be a lattice and let* $x, y, z \in L$. *Whenever the elements exist in* $\mathbf{L}$, *the following properties hold.*

(1) $x \leq y$ *if and only if* $y \dot{-} x = 0$.

(2) $y \dot{-} x \leq y$.

(3) $(y \vee z) \dot{-} x = (y \dot{-} x) \vee (z \dot{-} x)$.

(4) $z \dot{-} (x \wedge y) = (z \dot{-} x) \vee (z \dot{-} y)$.

(5) *If* $x \leq y$, *then* $x \dot{-} z \leq y \dot{-} z$ *and* $z \dot{-} x \geq z \dot{-} y$.

The filters corresponding to congruences were first characterised by Köhler [59], and Beazer [4] characterised subdirectly irreducible double Heyting algebras.

Definition 1.5.5. Let $\mathbf{A}$ be a double Heyting algebra and recall that, for all $x \in A$, we have ${\sim}x = 1 \dot{-} x$. We denote by $\mathbf{A}^\flat$ the H$^+$-algebra $\langle A; \vee, \wedge, \rightarrow, \sim, 0, 1\rangle$.

With the following theorem, Sankappanavar proved Beazer and Köhler's characterisations as corollaries of Theorem 1.4.4. Consequently, the theory of H$^+$-algebras subsumes much of the theory of double Heyting algebras.

Theorem 1.5.6 (Sankappanavar [79])**.** *Let* $\mathbf{A}$ *be a double Heyting algebra and let* $\theta \subseteq A^2$. *Then* θ *is a congruence on* $\mathbf{A}$ *if and only if* θ *is a congruence on* $\mathbf{A}^\flat$.

Corollary 1.5.7 (Köhler [59])**.** *Let* $\mathbf{A}$ *be a double Heyting algebra and let* F *be a filter of* $\mathbf{A}$. *Then* $\theta(F)$ *is a congruence if and only if* F *is closed under* $\neg{\sim}$.

Corollary 1.5.8 (Beazer [4])**.** *Let* $\mathbf{A}$ *be a double Heyting algebra of finite range. The following are equivalent*:

(1) $\mathbf{A}$ *is simple*;

(2) **A** *is subdirectly irreducible*;

(3) **A** *is directly indecomposable*;

(4) $\mathrm{Cen}(\mathbf{A}) = \{0, 1\}$.

Corollary 1.5.9 (Beazer [4])**.** *Let* **A** *be a double Heyting algebra.*

(1) **A** *is subdirectly irreducible if and only if there exists* $b \in A\backslash\{1\}$ *such that, for every* $a \in A\backslash\{1\}$, *there exists* $n \in \omega$ *such that* $(\neg{\sim})^n a \leq b$.

(2) **A** *is simple if and only if, for every* $a \in A\backslash\{1\}$, *there exists* $n \in \omega$ *such that* $d^n a = 0$.

In both cases, $\mathrm{Cen}(\mathbf{A}) = \{0, 1\}$.

Corollary 1.5.10 (Sankappanavar [79])**.** $\mathcal{DH}$ *has the congruence extension property.*

Corollary 1.5.11 (Sankappanavar [79])**.** *For each* $n \in \omega$, *the subvariety of* $\mathcal{DH}$ *defined by the identity* $(\neg{\sim})^{n+1}x = (\neg{\sim})^n x$ *is a discriminator variety.*

We will show later (Theorem 7.1.1) that the equation above completely characterises discriminator varieties of H^+-algebras and double Heyting algebras.

1.6 Regular double p-algebras

Definition 1.6.1. An algebra **A** is *congruence-regular* (or *regular* for short) if, for every $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$ and every $x \in A$,

$$x/\alpha = x/\beta \implies \alpha = \beta.$$

In other words, **A** is regular if, whenever two congruences share a class, they are actually the same congruence.

Theorem 1.6.2 (Varlet [88, 89], Katriňák [55])**.** *Let* **A** *be a double p-algebra. The following are equivalent*:

(1) **A** *is regular*;

(2) *for all* $x, y \in A$, *if* $\neg x = \neg y$ *and* ${\sim}x = {\sim}y$, *then* $x = y$;

(3) *every prime filter of* **A** *is minimal or maximal*;

(4) **A** *is distributive and* $\mathbf{A} \models {\sim}x \wedge x \leq y \vee \neg y$.

For the previous result, Varlet [88, 89] proved the equivalence of conditions (1), (2), and (3); condition (4) was included under the assumption of distributivity. Katriňák [55] extended this by proving that (2) implies distributivity. Notice that (4) provides an equational characterisation of regular double p-algebras.

Definition 1.6.3. Let $\mathcal{RDP}$ denote the variety of regular double p-algebras.

The following result of Katriňák shows that regular double p-algebras form a natural class of double Heyting algebras.

Theorem 1.6.4 (Katriňák [55])**.** *Let* $\mathbf{A}$ *be a congruence-regular double p-algebra. Then* $\mathbf{A}$ *is term-equivalent to a double Heyting algebra via the term*

$$x \to y = \neg\neg(\neg x \vee \neg\neg y) \wedge [{\sim}(x \vee \neg x) \vee \neg x \vee y \vee \neg y]$$

and its dual.

Thus, $\mathcal{RDP}$ is term-equivalent to a subvariety of $\mathcal{H}^+$ and $\mathcal{DH}$, so many of the results in this thesis will be applied to congruence-regular double p-algebras as well.

2

Topological duality

Priestley's duality for distributive lattices, first proved in [73], has been enormously useful in the study of distributive-lattice-based algebras. This thesis is no exception. The duality plays a minor role in the next two chapters, but it will offer its biggest contribution in Chapter 8. In this chapter, we will introduce our notation for ordered sets and Priestley duality, and we will cover some required preliminary results. We will state the restricted Priestley duality for Heyting algebras, H^+-algebras, and double Heyting algebras. In particular, we will detail the structure, the morphisms, and the operations. In Chapter 8, the duality is applied in a finite setting, so we also prove some properties of morphisms on finite ordered sets. We will assume the reader is familiar with Priestley's duality for bounded distributive lattices and will not dwell on the finer details. More information can be found in Davey and Priestley [26]. To ensure the reader is oriented correctly, note that the dual space is the set of prime filters and that the lattice is recovered by taking clopen upsets. Unless otherwise stated, the content of this chapter is entirely the original work of the author. The results are not yet published, but are currently included in a manuscript under preparation by Davey, Kowalski, and the author.

2.1 Ordered sets and fences

Definition 2.1.1. Let $\langle X; \leq \rangle$ be an ordered set and let $Y \subseteq X$. Define the following subsets of X:

(1) ${\uparrow}Y = \{x \in X \mid (\exists y \in Y)\ x \geq y\}$,

(2) ${\downarrow}Y = \{x \in X \mid (\exists y \in Y)\ x \leq y\}$,

(3) ${\updownarrow}Y = {\uparrow}Y \cup {\downarrow}Y$.

We will let $\leq_Y$ denote the order $\leq$ restricted to Y, that is, $\leq_Y\ = Y^2 \cap \leq$. Furthermore, Y is an *upset* if ${\uparrow}Y = Y$, and it is a *downset* if ${\downarrow}Y = Y$. Note too that there is a distinction between ${\updownarrow}Y$ and the sets ${\uparrow}{\downarrow}Y$ and ${\downarrow}{\uparrow}Y$. The set of minimal elements of

X will be denoted by $\min(X)$, and for each $Y \subseteq X$, we let $\min_X(Y) = \min(X) \cap Y$. Similarly, the set of maximal elements of X will be denoted by $\max(X)$, and for each $Y \subseteq X$, we let $\max_X(Y) = \max(X) \cap Y$. The lattice of upsets of X will be denoted by $\mathcal{U}(X)$, and the lattice of downsets by $\mathcal{O}(X)$. We will say that X is *connected* if, for all $x \in X$, there exists $n \in \mathbb{N}$ such that $\updownarrow^n x = X$.

In the study and application of ordered sets, a class of utmost importance is the class of fences. In the author's experience, fences are a rich source of counterexamples, and their use in Chapter 8 certainly testifies to this.

Definition 2.1.2. A non-trivial finite ordered set X is a *fence* if there is an enumeration $x_1, \ldots, x_n$ of elements of X, where $n = |X|$, such that the only order relations on X are given by one of the following:

(1) $x_1 < x_2 > x_3 < \cdots > x_{n-1} < x_n$,

(2) $x_1 < x_2 > x_3 < \cdots < x_{n-1} > x_n$, or

(3) $x_1 > x_2 < x_3 > \cdots > x_{n-1} < x_n$.

Examples of fences of each type are given in Figure 2.1. We will permit the two-element fence under this definition, which is covered by all of (1), (2), and (3). Note that, by assumption, a fence has at least two elements.

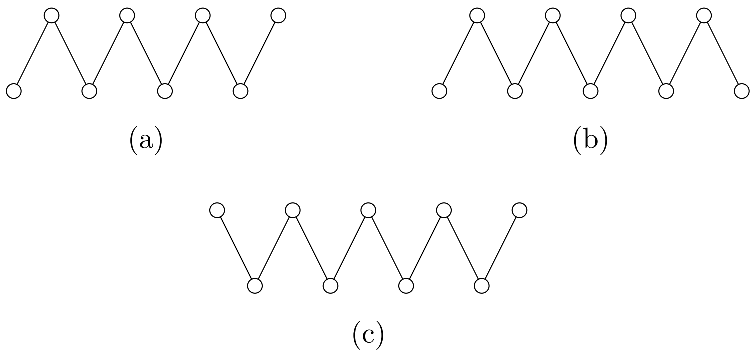


Figure 2.1: The fences (a), (b), and (c) are of type (1), (2), and (3) respectively.

This definition of a fence is not particularly user friendly, so we will give a characterisation that is more suited to the current setting.

Definition 2.1.3. Let X be an ordered set and let $\tau_1, \tau_2 \in X$ with $\tau_1 \neq \tau_2$. We will say that the pair (τ_1, τ_2) is an *up-tail* if τ_1 is maximal and ${\downarrow}\tau_1 = \{\tau_1, \tau_2\}$. Dually, (τ_1, τ_2) is a *down-tail* if τ_1 is minimal and ${\uparrow}\tau_1 = \{\tau_1, \tau_2\}$. In either case we will say that the pair (τ_1, τ_2) is a *tail* and that X *has a tail.*

(a) (b)

Figure 2.2: In (a), the pair (τ_1, τ_2) is an up-tail, and in (b), the pair (τ_1, τ_2) is a down-tail.

Observe that a tail (τ_1, τ_2) is both a down-tail and an up-tail if and only if $\updownarrow\{\tau_1, \tau_2\} = \{\tau_1, \tau_2\}$. Also note that if (τ_1, τ_2) is a down-tail, then τ_2 must be minimal, and if it is an up-tail, then τ_2 must be maximal.

Lemma 2.1.4. *Let X be a non-trivial finite connected ordered set. The following are equivalent:*

(1) *X is a fence;*

(2) *$|{\uparrow}x| \leq 3$ and $|{\downarrow}x| \leq 3$, for all $x \in X$, and if $|X| > 2$, then X has two tails;*

(3) *$|{\uparrow}x| \leq 3$ and $|{\downarrow}x| \leq 3$, for all $x \in X$, and X has at least one tail.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is obvious. As for (3) $\Rightarrow$ (1), we proceed by induction. If $|X| \in \{2, 3\}$, then the implication is obvious. So, assume that $|X| > 3$, that (3) holds for X, and that the characterisation holds for all fences of a smaller size than X. Let (x, y) be a tail in X. Assume first that (x, y) is a down-tail. Then x is minimal and y is maximal. Since X is connected and $|X| > 3$, there must be some $z \in X$ with $x \neq z$ such that $z \in {\downarrow}y$. Since $|{\downarrow}y| \leq 3$, we have that z is minimal and ${\downarrow}y = \{x, y, z\}$. Now consider the ordered set $Y = X \backslash \{x\}$, with the order inherited from X. Then, in Y, we have ${\downarrow}y = \{y, z\}$, so (y, z) is an up-tail in Y. Clearly all of the conditions in (3) hold for Y. Thus, Y is a fence, where the description of the order is of the form $\cdots < w > z < y$. Hence the order on X is of the form $\cdots < w > z < y > x$, and we conclude that X is a fence. A similar argument holds if we had instead assumed (x, y) to be an up-tail. $\square$

Remark 2.1.5. Since every element of a fence is either minimal or maximal, if X is a fence, then by Theorem 1.6.2, the lattice $\mathcal{U}(X)$ underlies a regular double p-algebra. Then by Theorem 1.6.4, up to term-equivalence, there is no difference between treating $\mathcal{U}(X)$ as a double p-algebra, an H^+-algebra, or a double Heyting algebra.

2.2 Duality for distributive lattices

Definition 2.2.1. Let $\mathbf{L}$ be a bounded distributive lattice and let $\mathcal{F}_p(\mathbf{L})$ denote the set of prime filters of $\mathbf{L}$. We will identify the set $\mathcal{F}_p(\mathbf{L})$ with the ordered topological space $\langle \mathcal{F}_p(\mathbf{L}); \subseteq, \mathcal{T} \rangle$, where the topology is generated by the sub-basis

$$\{X_a \mid a \in L\} \cup \{\mathcal{F}_p(\mathbf{L}) \backslash X_a \mid a \in L\},$$

with $X_a = \{F \in \mathcal{F}_p(\mathbf{L}) \mid a \in F\}$. A *Priestley space* is a structure $\langle X; \leq, \mathcal{T} \rangle$ such that $\langle X; \mathcal{T} \rangle$ is a compact topological space and, for all $x, y \in X$ with $x \nleq y$, there exists a clopen upset U such that $x \in U$ and $y \notin U$. The lattice of clopen upsets of a Priestley space X is denoted by $\mathcal{U}^{\mathcal{T}}(X)$, and the context will determine any further algebraic structure.

Priestley's duality establishes that the category of bounded distributive lattices with bounded lattice homomorphisms is dually equivalent to the category of Priestley spaces with continuous order-preserving maps. The properties in the next result will be used at various times without reference.

Proposition 2.2.2. *Let X be a Priestley space.*

(1) *The sets* $\min(X)$ *and* $\max(X)$ *are non-empty. Moreover, for all $x \in X$, both* $\min_X(\downarrow x)$ *and* $\max_X(\uparrow x)$ *are non-empty.*

(2) *Let Y and Z be disjoint closed subsets of X such that Y is an upset and Z is a downset. Then there exists a clopen upset W such that $Y \subseteq W$ and $W \cap Z = \varnothing$.*

(3) *If $\mathcal{U}^{\mathcal{T}}(X)$ is pseudocomplemented, then* $\max(X)$ *is closed, and if $\mathcal{U}^{\mathcal{T}}(X)$ is dually pseudocomplemented, then* $\min(X)$ *is closed.*

For (1) and (2), see Exercise 11.15 and Lemma 11.21 in [26]. A proof of (3) can be found in [75].

Definition 2.2.3. Let X be a Priestley space. Consider the following three conditions on X:

(P1) $\downarrow U$ is open, for every open set U in X,

(P2) $\uparrow U$ is open, for every open set U in X,

(P3) $\uparrow U$ is open, for every clopen downset U in X.

A Priestley space is a *Heyting space* if it satisfies (P1), an *H^+-space* if it satisfies (P1) and (P3), and a *double Heyting space* if it satisfies (P1) and (P2).

In [74], Priestley classified the dual spaces of distributive pseudocomplemented lattices and it was further elaborated on in [75]. The duality for Heyting algebras is generally attributed to Esakia [31] and often treated as folklore. A detailed exposition can be found in the appendix of [25]. Combining the results of those papers and dualising appropriately yields the next theorem.

Theorem 2.2.4. *Let X be a Priestley space. Then X is a Heyting space (resp. H^+-space, double Heyting space) if and only if $\mathcal{U}^{\mathcal{T}}(X)$ is the underlying lattice of a Heyting algebra (resp. H^+-algebra, double Heyting algebra).*

The next lemma summarises the operations.

Lemma 2.2.5. *Let X be a Priestley space and let $U, V \in \mathcal{U}^{\mathcal{T}}(X)$. If the corresponding operation is defined in $\mathcal{U}^{\mathcal{T}}(X)$, then*

(1) $\neg U = X\backslash{\downarrow}U$,

(2) ${\sim}U = {\uparrow}(X\backslash U)$,

(3) $U \to V = X\backslash{\downarrow}(U\backslash V)$,

(4) $U \dot{-} V = {\uparrow}(U\backslash V)$,

(5) $\neg{\sim}U = X\backslash{\downarrow}{\uparrow}(X\backslash U)$.

Of course, the duality is not complete without a description of the morphisms.

Definition 2.2.6. Let X and Y be Priestley spaces and let $\varphi\colon X \to Y$ be a continuous order-preserving map. We will then say that φ is a *morphism*. Consider the following three conditions on φ:

(M1) $(\forall x \in X)\ \varphi({\uparrow}x) = {\uparrow}\varphi(x)$,

(M2) $(\forall x \in X)\ \varphi({\downarrow}x) = {\downarrow}\varphi(x)$,

(M3) $(\forall x \in X)\ \varphi(\min_X({\downarrow}x)) = \min_Y({\downarrow}\varphi(x))$.

A morphism is a *Heyting morphism* if it satisfies (M1), an *H^+-morphism* if it satisfies (M1) and (M3), and a *double Heyting morphism* if it satisfies (M1) and (M2). For each $U \subseteq Y$, let $\varphi^{-1}(U) = \{x \in X \mid \varphi(x) \in U\}$.

Note that a double Heyting morphism is also an H$^+$-morphism. Also note that either of the conditions (M1) and (M2) on their own imply that the map is order-preserving, whereas (M3) is independent of this fact. Since we only apply condition (M3) in tandem with (M1) or (M2), the order-preserving assumption is redundant. By combining results from the papers cited earlier we obtain the next result.

Theorem 2.2.7. *Let X and Y be Priestley spaces and let $\varphi\colon X \to Y$ be a continuous map. Then φ is a Heyting morphism (resp. H^+-morphism, double Heyting morphism) if and only if the map $\varphi^{-1}\colon \mathcal{U}^{\mathcal{T}}(Y) \to \mathcal{U}^{\mathcal{T}}(X)$ is a Heyting algebra homomorphism (resp. H^+-algebra homomorphism, double Heyting algebra homomorphism).*

Definition 2.2.8. For convenience, we will often leave the codomain of a morphism implicit. If X and Y are Priestley spaces and $\varphi\colon X \to Y$ is a morphism, then we will say that φ is a *morphism on* X and let $\operatorname{codom}(\varphi) = Y$.

2.3 Morphisms and connected ordered sets

We now direct our focus to connected ordered sets. The results of this section will mainly be applied in Chapter 8. The proof of the next lemma is trivial.

Lemma 2.3.1. *Let X be an H^+-space, let φ be an H^+-morphism on X, and let $x \in X$. If x is maximal, then $\varphi(x)$ is maximal in* $\operatorname{codom}(\varphi)$*, and if x is minimal, then $\varphi(x)$ is minimal in* $\operatorname{codom}(\varphi)$.

Definition 2.3.2. We shall call a morphism φ on an ordered set X *degenerate* if it is a constant map or there exists $m_1 \in \max(X)$ and $m_2 \in \min(X) \setminus \max(X)$ such that $\varphi(m_1) = \varphi(m_2)$.

Henceforth, if φ is a morphism on a finite ordered set X and $S \subseteq X$, then we will identify $\varphi(S)$ with the ordered set $\langle \varphi(S); \leq_{\varphi(S)} \rangle$.

Lemma 2.3.3. *Let X be a connected ordered set and let φ be an H^+-morphism on X. Then φ is degenerate if and only if φ is a constant map.*

Proof. If φ is constant, then it is degenerate by definition. Conversely, assume φ is degenerate. It suffices to assume there exists a maximal $m_1 \in X$ and a minimal $m_2 \in X$ such that $\varphi(m_1) = \varphi(m_2)$. Let $m = \varphi(m_1) = \varphi(m_2)$. By Lemma 2.3.1, m is both minimal and maximal in $\varphi(X)$. But since X is connected, $\varphi(X)$ is also connected. This implies $\varphi(X) = \{m\}$, so φ is constant. □

Lemma 2.3.4. *Let X be an ordered set, let $x, y \in X$, and let φ be a non-degenerate H^+-morphism on X. If (τ_1, τ_2) is a down-tail in X, then $(\varphi(\tau_1), \varphi(\tau_2))$ is a down-tail in $\varphi(X)$.*

Proof. Assume that (τ_1, τ_2) is a down-tail. Then ${\uparrow}\varphi(\tau_1) = \varphi({\uparrow}\tau_1) = \{\varphi(\tau_1), \varphi(\tau_2)\}$. By Lemma 2.3.1, since τ_1 is minimal, $\varphi(\tau_1)$ is also minimal. The non-degeneracy assumption ensures $\varphi(\tau_1) \neq \varphi(\tau_2)$, and so $(\varphi(\tau_1), \varphi(\tau_2))$ is a down-tail. □

It is false that H^+-morphisms must preserve up-tails, although a dual argument to the one above shows that double Heyting morphisms do. This marks a notable distinction between the two types of morphism, and mildly complicates some of the proofs that follow. The proof of the next result is our motivation for the characterisation of fences in Lemma 2.1.4.

Proposition 2.3.5. *Let F be a fence and let φ be a non-degenerate H^+-morphism on F. Then $\langle \varphi(F); \leq_{\varphi(F)} \rangle$ is also a fence.*

Proof. Observe by Remark 2.1.5 that an H^+-morphism on F is a double Heyting morphism as well. So the dual of Lemma 2.3.4 applies. As F is connected, the image $\varphi(F)$ is also connected, and by using Lemma 2.3.4 and its dual we see that $\varphi(F)$ has at least one tail. For all $x \in F$, we have $|{\uparrow}x| \leq 3$, so $|{\uparrow}\varphi(x)| = |\varphi({\uparrow}x)| \leq 3$. Similarly, we have $|{\downarrow}\varphi(x)| \leq 3$. Thus, by Lemma 2.1.4, F is a fence. □

2.4 Double-pointed ordered sets

For this section, assume that every ordered set is finite.

Definition 2.4.1. A structure $\mathbf{S} = \langle S; \alpha, \beta, \leq \rangle$ is a *double-pointed ordered set* if $\langle S; \leq \rangle$ is a finite ordered set, α and β are nullary operations such that $\alpha^{\mathbf{S}} \neq \beta^{\mathbf{S}}$, and $\alpha^{\mathbf{S}}$ is minimal and $\beta^{\mathbf{S}}$ is maximal. We will use non-boldface lettering to denote the underlying ordered set. A map is a *morphism on* $\mathbf{S}$ if it is a morphism from the ordered set S to another (not necessarily double-pointed) ordered set.

The constraint that α is minimal and β is maximal is somewhat artificial, but we justify it for a few reasons. Although we can generalise some of the machinery below, the result we apply in Chapter 8, namely Corollary 2.4.11, is false if β is left arbitrary. We will also apply the results only with both α minimal and β maximal. Lastly, removing these constraints on α and β produces somewhat more cluttered proofs with no proportional increase in enlightenment. The next definition is essential.

Definition 2.4.2. Let $\mathbf{S}$ and $\mathbf{T}$ be double-pointed ordered sets and assume that $S \cap T = \varnothing$. Then $\mathbf{S} \searrow \mathbf{T}$ is the double-pointed ordered set $\langle S \cup T; \alpha, \beta, \leq^{\mathbf{S} \searrow \mathbf{T}} \rangle$ defined by

(1) $\leq^{\mathbf{S} \searrow \mathbf{T}} = \leq^{\mathbf{S}} \cup \leq^{\mathbf{T}} \cup \{(\alpha^{\mathbf{T}}, \beta^{\mathbf{S}})\}$,

(2) $\alpha^{\mathbf{S} \searrow \mathbf{T}} = \alpha^{\mathbf{S}}$,

(3) $\beta^{\mathbf{S} \searrow \mathbf{T}} = \beta^{\mathbf{T}}$.

Figure 2.3 illustrates the construction. It is easy to verify that $\mathbf{S} \searrow \mathbf{T}$ is an ordered set and that $\searrow$ is associative. To avoid excessive formality, we will always assume that different objects have disjoint sets—this is not unreasonable since we can just replace anything with an isomorphic copy of itself.

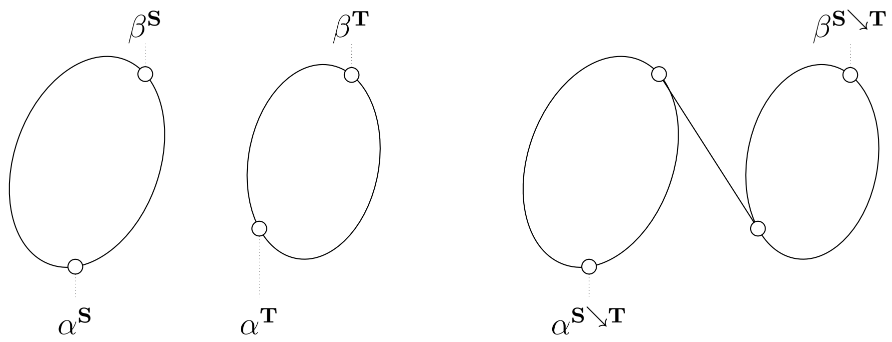


Figure 2.3: The double-pointed ordered sets $\mathbf{S}$ and $\mathbf{T}$ are on the left, and $\mathbf{S} \searrow \mathbf{T}$ is on the right.

If the ordered sets involved are connected, then the rightmost component $\mathbf{T}$ has a sort of absorbing behaviour under the image of an H^+-morphism on $\mathbf{S} \searrow \mathbf{T}$. Recall that if φ is a morphism on a finite ordered set X and $S \subseteq X$, then $\varphi(S)$ is identified with the ordered set $\langle \varphi(S); \leq_{\varphi(S)} \rangle$.

Lemma 2.4.3. *Let* $\mathbf{S}$ *and* $\mathbf{T}$ *be double-pointed ordered sets, assume* $\mathbf{S}$ *is connected, and let* φ *be an* H^+*-morphism on* $\mathbf{S} \searrow \mathbf{T}$*. If* $\varphi(\beta^{\mathbf{S}}) \in \varphi(T)$*, then* $\varphi(\mathbf{S} \searrow \mathbf{T}) = \varphi(T)$*.*

Proof. Assume that $\varphi(\beta^{\mathbf{S}}) \in \varphi(T)$ and let $x \in S$. By the connectedness of $\mathbf{S}$, every element of S is in the set $\updownarrow^n \beta^{\mathbf{S}}$, for some $n \in \omega$. We will prove that $\varphi(x) \in \varphi(T)$ implies $\varphi(\updownarrow x) \subseteq \varphi(T)$. The result will then follow by induction, as $\varphi(\beta^{\mathbf{S}}) \in \varphi(T)$. Let $y \in \updownarrow x$ and assume that $\varphi(x) \in \varphi(T)$. Then there is some $t \in T$ such that $\varphi(x) = \varphi(t)$. If $y \geq x$, then

$$\varphi(y) \in \varphi(\uparrow x) = \uparrow \varphi(x) = \uparrow \varphi(t) = \varphi(\uparrow t) \subseteq \varphi(T \cup \{\beta^{\mathbf{S}}\}),$$

which is a subset of $\varphi(T)$ by assumption. If $y \leq x$, then there is some minimal element $w \leq y$, and then, with $\mathbf{X} = \mathbf{S} \searrow \mathbf{T}$ and $Y = \mathrm{codom}(\varphi)$,

$$\varphi(w) \in \varphi(\min_X(\downarrow x)) = \min_Y(\downarrow \varphi(x)) = \min_Y(\downarrow \varphi(t)) = \varphi(\min_X(\downarrow t)).$$

So there is some $s \leq t$ such that $\varphi(w) = \varphi(s)$. Then, since $y \geq w$, we have

$$\varphi(y) \in \uparrow \varphi(w) = \uparrow \varphi(s) = \varphi(\uparrow s) \subseteq \varphi(T \cup \{\beta^{\mathbf{S}}\}) \subseteq \varphi(T),$$

as required. $\square$

Recall from Theorem 1.6.4 that a regular double p-algebra is term-equivalent to a double Heyting algebra. This implies that an H^+-morphism on the dual of a regular double p-algebra is also a double Heyting morphism. Since every prime filter in a regular double p-algebra is minimal or maximal, the next result slightly extends this observation.

Lemma 2.4.4. *Let* $\mathbf{S}$ *and* $\mathbf{T}$ *be double-pointed ordered sets and let* φ *be a non-degenerate* H^+*-morphism on* $\mathbf{S} \searrow \mathbf{T}$*. Assume that every element of* $\mathbf{T}$ *is minimal or maximal. Then, for all* $t \in T$*, we have* $\varphi({\downarrow}t) = {\downarrow}\varphi(t)$*. It follows that if* (x, y) *is an up-tail in* $\mathbf{T}$*, then* $(\varphi(x), \varphi(y))$ *is an up-tail in* $\varphi(\mathbf{S} \searrow \mathbf{T})$*.*

Proof. Let $\mathbf{X} = \mathbf{S} \searrow \mathbf{T}$, let $Y = \mathrm{codom}(\varphi)$, and let $t \in T$. If t is minimal, then $\varphi(t)$ is minimal, and the result holds trivially in that case. Assume that t is maximal. Then $\varphi(t)$ is maximal. Let $x \in X$ and assume $\varphi(x) \leq \varphi(t)$. Then there is some element $y \in X$ such that $\varphi(y)$ is minimal and $\varphi(y) \leq \varphi(x) \leq \varphi(t)$, implying $\varphi(y) \in \min_Y({\downarrow}\varphi(t)) = \varphi(\min_X({\downarrow}t))$. Therefore, there exists $w \in \min_X({\downarrow}t)$ such that $\varphi(y) = \varphi(w)$. So $\varphi(x) \in {\uparrow}\varphi(w) = \varphi({\uparrow}w)$, and since ${\uparrow}w \subseteq T \cup \{\beta^{\mathbf{S}}\}$, we must have that $\varphi(x)$ is minimal or maximal by assumption. Since $\varphi(t)$ is maximal and $\varphi(w) \leq \varphi(x) \leq \varphi(t)$, we conclude that $\varphi(x) \in \{\varphi(w), \varphi(t)\} \subseteq \varphi({\downarrow}t)$. It follows that ${\downarrow}\varphi(t) \subseteq \varphi({\downarrow}t)$, and the reverse inclusion holds because φ is order-preserving. To see that φ preserves up-tails in $\mathbf{T}$, use the dual of Lemma 2.3.4. □

The next result slightly extends Proposition 2.3.5.

Lemma 2.4.5. *Let* $\mathbf{S}$ *be a double-pointed ordered set, let* $\mathbf{F}$ *be a fence, and let* φ *be a non-degenerate* H^+*-morphism on* $\mathbf{S} \searrow \mathbf{F}$*. Then* $\varphi(F)$ *is a fence.*

Proof. We will use the characterisation of fences in Lemma 2.1.4. Since F is connected, so is $\varphi(F)$. If $|F| = 2$, then because φ is non-degenerate, it follows that $\varphi(F)$ is a connected ordered set with 2 elements, implying it is a two-element fence. If $|F| > 2$, then it is easy to see that $\mathbf{S} \searrow \mathbf{F}$ contains at least one tail. Specifically, the two tails of F are tails in $\mathbf{S} \searrow \mathbf{F}$, unless $\alpha^{\mathbf{F}}$ is the lower element of a down-tail, in which case the other tail in $\mathbf{F}$ is a tail in $\mathbf{S} \searrow \mathbf{F}$. Then, either by using Lemma 2.3.4 or Lemma 2.4.4, there is at least one tail in $\varphi(F)$. It only remains to check that $|\varphi(F) \cap {\downarrow}\varphi(x)| \leq 3$ and $|\varphi(F) \cap {\uparrow}\varphi(x)| \leq 3$, for all $x \in F$. Let $x \in F$. Since F is a fence, we have $|{\uparrow}x| \leq 3$, and then $\varphi({\uparrow}x) = {\uparrow}\varphi(x)$ implies $|\varphi(F) \cap {\uparrow}\varphi(x)| \leq 3$. Dually, by Lemma 2.4.4, we have $\varphi({\downarrow}x) = {\downarrow}\varphi(x)$, and so $|\varphi(F) \cap {\downarrow}\varphi(x)| \leq 3$. □

Definition 2.4.6. A double-pointed ordered set $\mathbf{T}$ is an *ordered set with a down-tail* if there exists $\tau_1, \tau_2 \in T$ such that $\alpha^{\mathbf{T}} = \tau_1$ and (τ_1, τ_2) is a down-tail. In what follows, we will let $\tau_1^{\mathbf{T}}$ and $\tau_2^{\mathbf{T}}$ denote τ_1 and τ_2 as stated here.

We draw special attention to the fact that, according to the above definition, if it is specified that a double-pointed ordered set $\mathbf{T}$ has a down-tail, then we are assuming that $\alpha^{\mathbf{T}}$ is part of that down-tail. In that case, $\searrow$ entails a more specific construction (see Figure 2.4). However, $\beta^{\mathbf{T}}$ is still an arbitrary maximal.

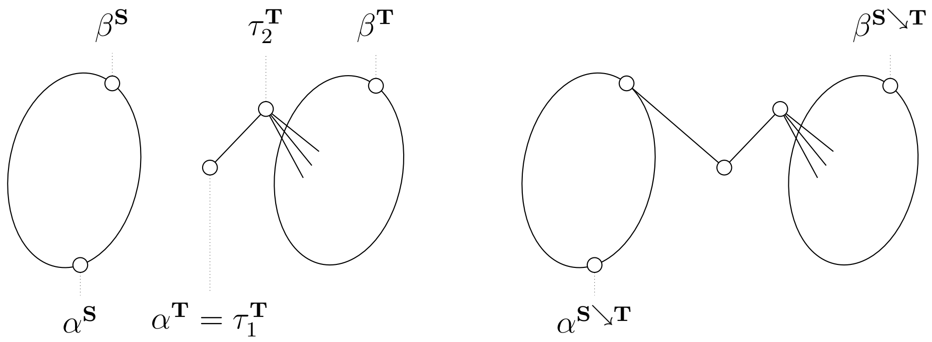


Figure 2.4: Special case: $\mathbf{S} \searrow \mathbf{T}$ when $\mathbf{T}$ has a specified down-tail.

Lemma 2.4.7. *Let* $\mathbf{S}$ *be a double-pointed ordered set, let* $\mathbf{T}$ *be an ordered set with a down-tail, and let* φ *be an* H^+*-morphism on* $\mathbf{S} \searrow \mathbf{T}$*. If* $\varphi(\beta^{\mathbf{S}}) \notin \varphi(T)$*, then* $\varphi(\tau_2^{\mathbf{T}}) \notin \varphi(T \backslash \{\tau_2\})$*.*

Proof. Let $\tau_1 = \tau_1^{\mathbf{T}}$, let $\tau_2 = \tau_2^{\mathbf{T}}$, let $\mathbf{X} = \mathbf{S} \searrow \mathbf{T}$, and let $Y = \operatorname{codom}(\varphi)$. Assume that $\varphi(\beta^{\mathbf{S}}) \notin \varphi(T)$. Suppose, by way of contradiction, that there is some $t \in T\backslash\{\tau_2\}$ such that $\varphi(t) = \varphi(\tau_2)$. Note that $\tau_1 \notin {\downarrow}t$ because $t \neq \tau_2$. But since τ_1 is minimal, we have $\varphi(\tau_1) \in \min(Y)$. Then since φ is order-preserving, we have $\varphi(\tau_1) \leq \varphi(\tau_2)$. Thus,

$$\varphi(\tau_1) \in \min_Y({\downarrow}\varphi(\tau_2)) = \min_Y({\downarrow}\varphi(t)) = \varphi(\min_X({\downarrow}t)).$$

So there exists $s \in {\downarrow}t$ such that $\varphi(s) = \varphi(\tau_1)$ and $s \neq \tau_1$. Note that ${\uparrow}s \subseteq T$ because $s \neq \tau_1$. By construction, we have $\beta^{\mathbf{S}} > \alpha^{\mathbf{T}} = \tau_1$, and then because $\varphi({\uparrow}\tau_1) = {\uparrow}\varphi(\tau_1) = {\uparrow}\varphi(s) = \varphi({\uparrow}s)$, it follows that there must be some $u \in {\uparrow}s$ such that $\varphi(\beta^{\mathbf{S}}) = \varphi(u)$. By assumption, u cannot be in T, but $u \in {\uparrow}s \subseteq T$, a contradiction. □

The final leg of this section returns the focus to fences.

Definition 2.4.8. Let $\mathbf{X}$ be an ordered set with a down-tail. If the underlying ordered set of $\mathbf{X}$ is a fence, then we say that $\mathbf{X}$ is a *fence with a down-tail.*

From Lemma 2.4.3 and Lemma 2.4.5 we obtain the next result.

Lemma 2.4.9. *Let* $\mathbf{S}$ *be a connected double-pointed ordered set, let* $\mathbf{F}$ *be a fence with a down-tail, and let* φ *be a non-degenerate* H^+*-morphism on* $\mathbf{S}\searrow\mathbf{F}$*. If* $\varphi(\beta^{\mathbf{S}}) \in \varphi(F)$*, then* $\varphi(\mathbf{S} \searrow \mathbf{F})$ *is a fence.*

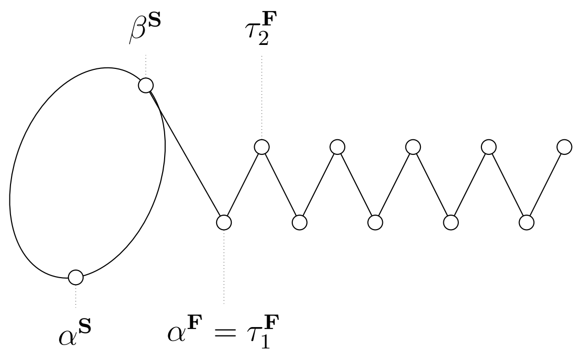


Figure 2.5: An example of an ordered set of the form $\mathbf{S} \searrow \mathbf{F}$, where $\mathbf{F}$ is a fence with a down-tail. Note that $\alpha^{\mathbf{S}\searrow\mathbf{F}} = \alpha^{\mathbf{S}}$ and $\beta^{\mathbf{S}\searrow\mathbf{F}} = \beta^{\mathbf{F}}$, the latter of which has been omitted from the diagram.

We will denote the restriction of a map φ to a set X by $\varphi\restriction_X$.

Lemma 2.4.10. *Let $\mathbf{S}$ be a connected ordered set, let $\mathbf{F}$ be a fence with a down-tail, and let φ be a non-degenerate H^+-morphism on $\mathbf{S} \searrow \mathbf{F}$. If $\varphi(\beta^{\mathbf{S}}) \notin \varphi(F)$, then $\varphi\restriction_F$ is one-to-one.*

Proof. Assume $\varphi(\beta^{\mathbf{S}}) \notin \varphi(F)$. If $|F| = 2$, the result holds because φ is non-degenerate. Assume $|F| \geq 3$. Then there exists $\gamma \in F$ such that ${\downarrow}\tau_2^{\mathbf{F}} = \{\tau_1^{\mathbf{F}}, \tau_2^{\mathbf{F}}, \gamma\}$. If $|F| = 3$, it needs only to be checked that $\varphi(\tau_1^{\mathbf{F}}) \neq \varphi(\gamma)$. But since ${\uparrow}\gamma = \{\gamma, \tau_2^{\mathbf{F}}\}$ and $\beta^{\mathbf{S}} > \tau_1^{\mathbf{F}}$, if $\varphi(\tau_1^{\mathbf{F}}) = \varphi(\gamma)$, then $\varphi(\beta^{\mathbf{S}}) \in {\uparrow}\varphi(\tau_1^{\mathbf{F}}) = {\uparrow}\varphi(\gamma) \subseteq \varphi(F)$, a contradiction. So $\varphi(\tau^{\mathbf{F}}) \neq \varphi(\gamma)$. Let $|F| > 3$ and assume inductively that the result holds for all fences of a smaller size. It is easy to see that $F' = F \backslash \{\tau_1^{\mathbf{F}}, \tau_2^{\mathbf{F}}\}$ is a fence and γ is the minimum element of a down-tail in F'. Let $\mathbf{F}'$ be a fence with a down-tail such that its underlying ordered set is F' and $\alpha^{\mathbf{F}'} = \gamma$, with $\beta^{\mathbf{F}'}$ left arbitrary. Define $\mathbf{T}$ on $T = \{\tau_1^{\mathbf{F}}, \tau_2^{\mathbf{F}}\}$ by $\alpha^{\mathbf{T}} = \tau_1^{\mathbf{F}}$ and $\beta^{\mathbf{T}} = \tau_2^{\mathbf{F}}$. Then the underlying ordered sets of $\mathbf{F}$ and $\mathbf{T} \searrow \mathbf{F}'$ are equal. Thus, the underlying ordered sets of $\mathbf{S} \searrow \mathbf{F}$ and $(\mathbf{S} \searrow \mathbf{T}) \searrow \mathbf{F}'$ are also equal. Since $\varphi(\beta^{\mathbf{S}}) \notin \varphi(F)$, it follows by Lemma 2.4.7 that $\varphi(\tau_2^{\mathbf{F}}) \notin \varphi(F')$. So by the inductive hypothesis, φ is one-to-one on F'. It remains to show that $\varphi(\tau_1^{\mathbf{F}}) \notin \varphi(F')$. But if this were the case, since $\beta^{\mathbf{S}} > \tau_1^{\mathbf{F}}$, we would have $\varphi(\beta^{\mathbf{S}}) \in \varphi(F)$, a contradiction. □

This final corollary is the key result used in Section 8.2.

Corollary 2.4.11. *Let $\mathbf{S}$ be a connected double-pointed ordered set, let $\mathbf{F}$ be a fence with a down-tail, and let φ be a non-degenerate H^+-morphism on $\mathbf{S} \searrow \mathbf{F}$. If $\varphi\restriction_F$ is not one-to-one, then $\varphi(\mathbf{S} \searrow \mathbf{F})$ is a fence.*

Proof. By Lemma 2.4.10, if $\varphi\restriction_F$ is not one-to-one, then $\varphi(\beta^{\mathbf{S}}) \in \varphi(F)$, and then $\varphi(\mathbf{S} \searrow \mathbf{F})$ is a fence by Lemma 2.4.9. □

3

Properties and examples

In this chapter, we give some preliminary results and supply a few natural classes of double Heyting algebras in the wild. We first create some examples to confirm that H^+-algebras and double Heyting algebras are genuinely different algebraic structures. Some general criteria guaranteeing that a lattice is the underlying lattice of a double Heyting algebra are given. We also consider splittings in a lattice, which have proved to be quite important in the study of lattices of subvarieties. An argument courtesy of Brian Davey shows that every covering pair in a double Heyting algebra induces a splitting of the lattice. Another argument by Brian Davey proves that the lattice of subvarieties of a locally finite congruence-distributive variety forms a double Heyting algebra. Apart from these subvariety lattices, there are various other interesting occurrences of double Heyting algebras. For example, Ghilardi [36] proved that every free Heyting algebra on finitely many generators is the reduct of a double Heyting algebra. This was extended further by Butz [18] who showed that finitely presented Heyting algebras also form double Heyting algebras. In neither case is the dual implication a term in the language of Heyting algebras.

We also consider more concrete examples. For instance, it is easy to show that the lattice of open sets of a topological space forms a Heyting algebra, and a natural question is to follow that up in the dual. We have also found examples of double Heyting algebras in graph theory. In Section 3.5 we see that the lattice of subgraphs of a graph forms a double Heyting algebra. The converse is tackled in Chapter 4. Another graph-theoretical example is an ordered set obtained by considering homomorphisms on finite directed graphs. It is well known that the graph homomorphism lattice, which we define in Section 3.6, is a Heyting algebra, and it is famous for its complexity: every countable ordered set embeds into it. We use an argument by Brian Davey to prove that it is also a double Heyting algebra. This result is not known in the literature. The results of this chapter have not been published, but those attributed to Davey are currently included in a joint manuscript under preparation by Davey, Kowalski, and the author.

3.1 General properties and abstract examples

For any class $\mathcal{K}$ of lattice-based algebras, we say that a lattice $\mathbf{L}$ *forms* an algebra in $\mathcal{K}$ if there is an algebra in $\mathcal{K}$ whose lattice reduct is $\mathbf{L}$. The first result we use follows easily from the definitions and provides a constructive description of the operations under consideration. For reference, see Balbes and Dwinger [2].

Lemma 3.1.1. *Let* $\mathbf{L}$ *be a lattice and let* $x, y \in L$. *Whenever the elements exist in* $\mathbf{L}$*, the following equations hold*:

$$\begin{aligned} \neg x &= \max\{z \in L \mid x \wedge z = 0\}, \\ x \to y &= \max\{z \in L \mid x \wedge z \leq y\}, \\ {\sim} x &= \min\{z \in L \mid x \vee z = 1\}, \\ y \dot{-} x &= \min\{z \in L \mid x \vee z \geq y\}. \end{aligned}$$

If one side of an equation above exists, then so does the other.

Immediately following from this is our first example.

Example 3.1.2. Let $\mathbf{L}$ be a complete, completely distributive lattice. Then $\mathbf{L}$ forms both an H^+-algebra and a double Heyting algebra. In particular, every finite distributive lattice forms an H^+-algebra and a double Heyting algebra. More generally, any lattice satisfying (JID) forms a Heyting algebra, and any lattice satisfying (MID) forms a dual Heyting algebra; hence, a lattice satisfying both (JID) and (MID) forms a double Heyting algebra.

In fact, (JID) characterises complete Heyting algebras.

Lemma 3.1.3. *Let* $\mathbf{L}$ *be a complete lattice. Then* $\mathbf{L}$ *forms a Heyting algebra if and only if* $\mathbf{L}$ *satisfies* (JID).

The fact that every finite distributive lattice forms an H^+-algebra and a double Heyting algebra is crucial to the next proof.

Theorem 3.1.4. *The varieties* $\mathcal{DH}$ *and* $\mathcal{H}^+$ *are generated by their finite members.*

Proof. We prove that if an identity fails in a double Heyting algebra, then it fails in a finite double Heyting algebra. Since H^+-algebra terms are also double Heyting algebra terms, the same argument shows that if an identity fails in an H^+-algebra, then it fails in a finite H^+-algebra. Let $\mathbf{A}$ be a double Heyting algebra and assume the identity $s = t$ fails in $\mathbf{A}$. Then there exists a tuple $\overline{a}$ of elements of A such that $s^{\mathbf{A}}(\overline{a}) \neq t^{\mathbf{A}}(\overline{a})$. Let Σ be the set of terms that are subterms of s or t. Note that Σ is finite. Let $\mathbf{B}$ be the *sublattice* of $\mathbf{A}$ generated by the set $\{\sigma^{\mathbf{A}}(\overline{a}) \mid \sigma \in \Sigma\}$. Then $\mathbf{B}$ is a finite distributive lattice, so it forms a double Heyting algebra. By treating $\mathbf{B}$ as a double Heyting algebra, we have $s^{\mathbf{B}}(\overline{a}) = s^{\mathbf{A}}(\overline{a})$ by construction, and similarly for t, so the identity $s = t$ fails in $\mathbf{B}$ as well. □

Open Problem 1. Is the variety of regular double p-algebras generated by its finite members?[1] The proof of Theorem 3.1.4 will not apply to regular double p-algebras. This is because, by Theorem 1.6.2, the underlying lattice of a finite regular double p-algebra must have no chain of three or more join-irreducible elements, and that is not a property preserved by sublattices.

The similarity of the algebras we consider prompts two questions. Firstly, does every H$^+$-algebra form a double Heyting algebra? Secondly, if an H$^+$-algebra $\mathbf{A}$ also forms a double Heyting algebra, is $\dot{-}$ definable on $\mathbf{A}$ as a $\langle \vee, \wedge, \to, \sim, 0, 1\rangle$-term function? The answer to both questions is negative. By Example 3.1.2, any counterexample to the first question must be infinite. This is best exhibited using the topological duality.

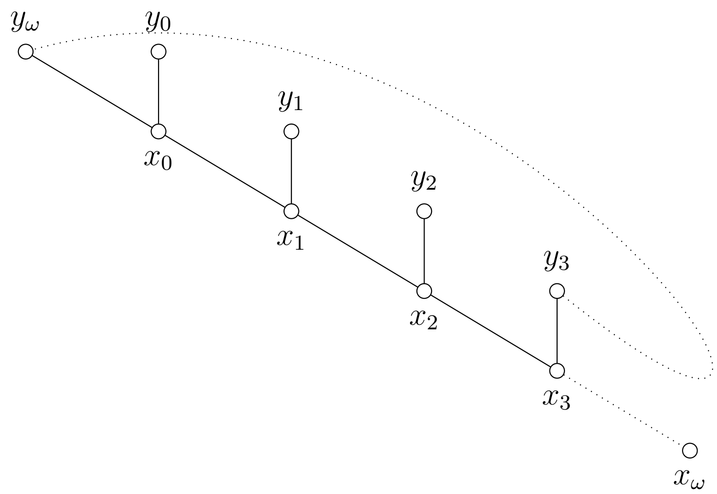


Figure 3.1: An H$^+$-space that is not a double Heyting space.

Example 3.1.5. Let $X = \{x_i \mid i \in \omega\}$ and $Y = \{y_i \mid i \in \omega\}$ be disjoint countable sets endowed with the discrete topology. Let $X^* = X \cup \{x_\omega\}$ be the one-point compactification of X. A subset $U \subseteq X^*$ is open in X^* if and only if either

(1) $x_\omega \notin U$, or

(2) $x_\omega \in U$ and $X \backslash U$ is finite.

[1]Added in proof: Tomasz Kowalski has very recently announced a proof that the variety of regular double p-algebras is generated by its finite members.

Define Y^* similarly. Now let $Z = X^* \cup Y^*$, and declare a set $U \subseteq Z$ open if and only if $U \cap X^*$ and $U \cap Y^*$ are open in X^* and Y^* respectively. Define the order on Z as given in Figure 3.1. Since every non-empty downset in Z contains x_ω, it is easily verified according to Definition 2.2.3 that Z is an H^+-space. To see that it is not a double Heyting space, observe that the set $\{x_0\}$ is open but $\uparrow\{x_0\} = \{x_0, y_0, y_\omega\}$ is not. Thus, $\mathcal{U}^{\mathcal{T}}(Z)$ forms an H^+-algebra but not a double Heyting algebra.

For the second question, a finite counterexample suffices.

Example 3.1.6. Let $\mathbf{L}$ be the lattice shown in Figure 3.2. The shaded elements depict a $\langle \vee, \wedge, \to, \sim, 0, 1\rangle$-subuniverse of $\mathbf{L}$ that is not closed under $\dot{-}$ because it does not contain $y = x \dot{-} {\sim}{\sim}x$. It follows that $\dot{-}$ cannot be defined on $\mathbf{L}$ in the language $\langle \vee, \wedge, \to, \sim, 0, 1\rangle$.

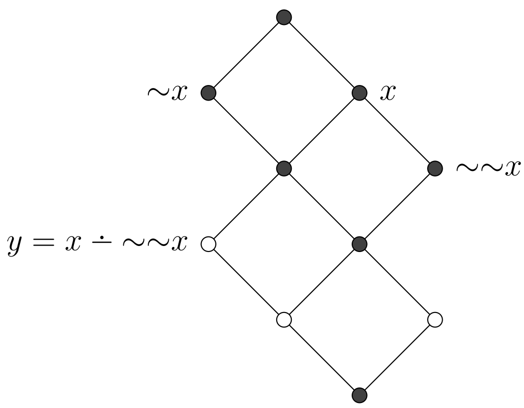


Figure 3.2

From Theorem 1.6.2, if $\mathbf{A}$ is an H^+-algebra and there are no chains of three or more elements in the dual space, then $\mathbf{A}$ forms a regular double p-algebra. Then it is term-equivalent to a double Heyting algebra by Theorem 1.6.4. The lattices of Example 3.1.5 and 3.1.6 do not form regular double p-algebras. The first has infinite chains of prime filters, and the second has chains of prime filters of size three. All of the examples we know of H^+-algebras that do not form double Heyting algebras have infinite chains of prime filters. Thus, we raise the next question.

Open Problem 2. Let $n \in \omega$, let $\mathbf{A}$ be an H^+-algebra, and assume every chain of prime filters of $\mathbf{A}$ has at most n elements. Does the underlying lattice of $\mathbf{A}$ form a double Heyting algebra? More generally, what conditions can ensure a Heyting algebra or an H^+-algebra also forms a double Heyting algebra?

Lemma 3.1.8 below, which was proved by Brian Davey, partially answers this question by supplying a simple sufficient condition for a lattice to form a dual Heyting algebra. It will be quite useful in the upcoming sections.

Definition 3.1.7. Let $\mathbf{L}$ be a lattice and let $x \in L$. Then x is *join-prime* provided that, for all $a, b \in L$, if $a \vee b \geq x$, then $a \geq x$ or $b \geq x$. If $\mathbf{L}$ is complete, then x is *completely join-prime* provided that, for every $A \subseteq L$, if $\bigvee A \geq x$, then there is some $a \in A$ such that $a \geq x$. A set $P \subseteq L$ is *join-dense in* $\mathbf{L}$ if every element in L is the join in $\mathbf{L}$ of some subset of P. The same notions for meets are defined dually.

Lemma 3.1.8. *Let* $\mathbf{L}$ *be a lattice with a bottom in which every element is the join of a finite set of join-prime elements. Then* $y \dot{-} x$ *exists, for all* $x, y \in L$.

Proof. Since $y \dot{-} x = (x \vee y) \dot{-} x$, it is sufficient to prove that $y \dot{-} x$ exists in $\mathbf{L}$ whenever $y \geq x$. Let $x, y \in L$ and assume $y \geq x$. By assumption, there are finite sets X and Y of join-prime elements such that $x = \bigvee X$ and $y = \bigvee Y$. Now define the sets F_x and F_z by $F_x = \{a \in X \cup Y \mid a \leq x\}$ and $F_z = \{a \in X \cup Y \mid a \nleq x\}$. We have $x = \bigvee F_x$ and $y = \bigvee (F_x \cup F_z)$. Let $z = \bigvee F_z$, which exists because F_z is finite. We claim that $y \dot{-} x = z$. We will prove that, for all $a \in L$,

$$a \vee x \geq y \iff a \geq z,$$

and then the result holds by definition of $\dot{-}$. Let $a \in L$. First, assume $a \geq z$. Then

$$a \vee x \geq z \vee x = \bigvee F_z \vee \bigvee F_x = \bigvee (F_z \cup F_x) = y.$$

Conversely, assume $a \vee x \geq y$ and let $b \in F_z$. Then by definition, we have $y \geq b$, and so $a \vee x \geq b$. Since b is join-prime and $x \ngeq b$, we have $a \geq b$. Hence a is an upper bound of F_z, so it follows that $a \geq z$, as required. □

3.2 Splittings

Splittings for lattices were first introduced by Whitman in [92]. In this section we will investigate splittings and their relationship with double Heyting algebras.

Definition 3.2.1. Let $\mathbf{L}$ be a lattice and let $a, b \in L$. Then (a, b) is called a *splitting pair* if $a \nleq b$ and ${\uparrow}a \cup {\downarrow}b = L$.

Proving the following two lemmas is completely straightforward.

Lemma 3.2.2. *Let* $\mathbf{L}$ *be a complete lattice and let* $a, b \in L$. *The following are equivalent*:

(1) (a, b) *is a splitting pair*;

(2) a *is completely join-prime and* $b = \bigvee \{x \in L \mid a \nleq x\}$;

(3) b *is completely meet-prime and* $a = \bigwedge \{x \in L \mid x \nleq b\}$.

Lemma 3.2.3. *Let* $\mathbf{L}$ *be a lattice and let* $a, b \in L$*. If* (a, b) *is a splitting pair, then*

(1) $a \wedge b \prec a$,

(2) $b \prec a \vee b$,

(3) $a \to (a \wedge b)$ *exists and is equal to* b,

(4) $(a \vee b) \dot{-} b$ *exists and is equal to* a.

Note that the above lemma does not imply $x \to y$ and $x \dot{-} y$ are always defined. We will prove a sort of converse to the previous lemma. The next three results and their proofs are due to Brian Davey.

Lemma 3.2.4. *Let* $\mathbf{L}$ *be a lattice and let* $a, b \in L$*. Assume that* $a \prec b$ *and that* $b \to a$ *exists in* $\mathbf{L}$*. For all* $x \in L$*, if* $x \vee a \not\leq b$*, then* $x \leq b \to a$*.*

Proof. Let $x \in L$ and assume $x \vee a \not\leq b$. Then $(x \vee a) \wedge b < b$. Since $a \prec b$ and $a \leq x \vee a$, we have $a \leq (x \vee a) \wedge b < b$, and so $a = (x \vee a) \wedge b$ by assumption. By definition of $\to$, we then have $x \vee a \leq b \to a$, and the result holds because $x \leq x \vee a$. □

Lemma 3.2.5. *Let* $\mathbf{L}$ *be a lattice and let* $a, b, c \in L$*. Assume that* $a \prec b$ *and that* $b \to a$ *exists in* $\mathbf{L}$*. If* c *is join-prime,* $c \leq b$*, and* $c \not\leq a$*, then* $(c, b \to a)$ *is a splitting pair in* $\mathbf{L}$*.*

Proof. Assume that c is join-prime, $c \leq b$, and $c \not\leq a$. First, suppose that $c \leq b \to a$. Then by definition of $\to$, we have $c \wedge b \leq a$, but $c \wedge b = c$ and $c \not\leq a$, a contradiction. So $c \not\leq b$. It remains to show that $L = {\uparrow} c \cup {\downarrow}(b \to a)$. Let $x \in L$ and assume $x \not\geq c$. Since $a \not\geq c$ and c is join-prime, we have $a \vee x \not\geq c$. Since $b \geq c$, it follows that $a \vee x \not\geq b$. Then by Lemma 3.2.4, we have $x \leq b \to a$, as required. □

If both $b \dot{-} a$ and $b \to a$ exist, then we can relax the assumptions of the previous lemma.

Lemma 3.2.6. *Let* $\mathbf{L}$ *be a lattice and let* $a, b \in L$*. Assume that* $a \prec b$ *and that both* $b \dot{-} a$ *and* $b \to a$ *exist in* $\mathbf{L}$*. Then* $(b \dot{-} a, b \to a)$ *is a splitting pair in* $\mathbf{L}$*. It follows that* $b \dot{-} a$ *is completely join-prime and* $b \to a$ *is completely meet-prime.*

Proof. Firstly,

$$
\begin{aligned}
b \dot{-} a \leq b \to a &\iff b \leq a \vee (b \to a) && \text{by definition of } \dot{-} \\
&\iff b \leq b \to a && \text{because } a \leq b \to a \\
&\iff b \wedge b \leq a && \text{by definition of } \to \\
&\iff b \leq a.
\end{aligned}
$$

Since $a \prec b$, we conclude that $b \dot{-} a \nleq b \to a$. All that remains is to show that $L = {\uparrow}(b \dot{-} a) \cup {\downarrow}(b \to a)$. Let $x \in L$ and assume $x \ngeq b \dot{-} a$. Then by definition, we have $x \vee a \ngeq b$, and so $x \leq b \to a$ by Lemma 3.2.4. This completes the proof. □

Splittings have been of great interest in the study of lattices of subvarieties.

Definition 3.2.7. We will denote the lattice of subvarieties of a variety $\mathcal{V}$ by $\mathcal{L}(\mathcal{V})$.

The following lemma follows easily by using Lemma 3.2.2 and the fact that every variety is generated by its finitely generated subdirectly irreducible members.

Lemma 3.2.8. *Let $\mathcal{V}$ be a variety and let $(\mathcal{A}, \mathcal{B})$ be a splitting pair in $\mathcal{L}(\mathcal{V})$. Then there is a finitely generated subdirectly irreducible algebra $\mathbf{A}$ such that $\mathcal{A} = \mathrm{Var}(\mathbf{A})$, and $\mathcal{B}$ is defined by a single equation relative to $\mathcal{V}$.*

This motivates the next definition.

Definition 3.2.9. Let $\mathcal{V}$ be a variety and let $\mathbf{A}$ be a finitely generated subdirectly irreducible algebra in $\mathcal{V}$. If there exists $\mathcal{B} \in \mathcal{L}(\mathcal{V})$ such that $(\mathrm{Var}(\mathbf{A}), \mathcal{B})$ is a splitting pair in $\mathcal{L}(\mathcal{V})$, then we say that $\mathbf{A}$ is a *splitting algebra* (*in* $\mathcal{V}$).

As an example, Jankov [50] proved that every finite subdirectly irreducible Heyting algebra is a splitting algebra in the variety of Heyting algebras, although using a different (but equivalent) approach without reference to splittings. For the historical notes, refer to Section 9.8 and Theorem 10.46 in [23]. A result of Blok and Pigozzi [10, Corollary 3.2] generalises Jankov's result, which shows that if $\mathbf{A}$ is a finite subdirectly irreducible algebra in a variety with a finite signature and equationally definable principal congruences, then $\mathbf{A}$ is a splitting algebra. McKenzie [69] studied splittings in the lattice of subvarieties of lattices. Many techniques for splitting algebras were introduced in [69]. In particular, McKenzie proved that every splitting algebra in the variety of lattices is finite. The proof is easily generalised to the following result.

Lemma 3.2.10. *If $\mathcal{V}$ is a congruence-distributive variety and is generated by its finite members, then every splitting algebra in $\mathcal{V}$ is finite.*

A modification of Jankov's technique was used by Kowalski and Ono [63] to prove that the only splitting algebra in the variety of $\mathrm{FL_{ew}}$ algebras is the two-element Boolean algebra. We will use a modification of their modification in Section 5.5.

3.3 Locally finite congruence-distributive varieties

For any variety $\mathcal{V}$, the lattice of subvarieties $\mathcal{L}(\mathcal{V})$ is a dually algebraic lattice. Then every dually compact element in such a lattice is part of a covering pair. Thus, if a subvariety lattice forms a double Heyting algebra, then by Lemma 3.2.6, it is also rich with splittings. If $\mathcal{V}$ is congruence-distributive, then $\mathcal{L}(\mathcal{V})$ satisfies (MID), and so, by Example 3.1.2, it forms a dual Heyting algebra. Moreover, if $\mathcal{L}(\mathcal{V})$ is algebraic, then it also satisfies (JID), in which case it forms a double Heyting algebra. Hence, if $\mathcal{V}$ is a congruence-distributive variety and $\mathcal{L}(V)$ is algebraic, then $\mathcal{L}(V)$ forms a double Heyting algebra.

In this section, we will prove a sufficient condition that ensures $\mathcal{L}(\mathcal{V})$ is algebraic. Although we are unaware of any internal characterisation of algebraic subvariety lattices, by using [26, Theorem 10.29] and Lemma 3.1.3, the following theorem characterises them in terms of the lattice itself.

Theorem 3.3.1. *Let $\mathcal{V}$ be a variety. The following are equivalent*:

(1) $\mathcal{L}(\mathcal{V})$ *forms a Heyting algebra* (*and therefore a double Heyting algebra*);

(2) $\mathcal{L}(\mathcal{V})$ *satisfies* (JID);

(3) $\mathcal{L}(\mathcal{V})$ *is distributive and algebraic*;

(4) $\mathcal{L}(\mathcal{V})$ *is completely distributive*;

(5) *every completely join-irreducible element of* $\mathcal{L}(\mathcal{V})$ *is completely join-prime*;

(6) *the completely join-prime elements of* $\mathcal{L}(\mathcal{V})$ *are join-dense in* $\mathcal{L}(\mathcal{V})$;

(7) $\mathcal{L}(\mathcal{V})$ *is isomorphic to* $\mathcal{O}(P)$ *via* $\mathcal{A} \mapsto \{\mathcal{B} \in P \mid \mathcal{B} \subseteq \mathcal{A}\}$, *where P is the ordered set of subvarieties of $\mathcal{V}$ that are completely join-prime in* $\mathcal{L}(\mathcal{V})$.

Open Problem 3. Find an algebraic description of varieties whose subvariety lattices form a Heyting algebra.

We will now use an argument by Brian Davey to prove that the lattice of subvarieties of a locally finite variety is algebraic.

Definition 3.3.2. Let $\mathcal{V}$ be a variety of any signature. Then $\mathcal{V}$ is *locally finite* if every finitely generated algebra in $\mathcal{V}$ is finite, and the variety is *finitely generated* if $\mathcal{V} = \mathrm{Var}(\mathbf{A})$, for some finite algebra $\mathbf{A}$.

Lemma 3.3.3. *Let $\mathcal{V}$ be a locally finite variety. Then $\mathcal{L}(\mathcal{V})$ is an algebraic lattice. A subvariety of $\mathcal{V}$ is compact if and only if it is finitely generated.*

Proof. The first claim follows from the second. To see this, first recall that every variety is generated by the finitely generated algebras it contains. Since $\mathcal{V}$ is locally finite, every finitely generated algebra in $\mathcal{V}$ is finite. Thus, if $\mathcal{A}$ is a subvariety of $\mathcal{V}$, then it is equal to the join of its finitely generated subvarieties.

Now we prove the second claim. Let $\mathcal{A}$ be a subvariety of $\mathcal{V}$. Then by the previous paragraph,

$$\mathcal{A} = \bigvee\{\mathcal{B} \in \mathcal{L}(\mathcal{V}) \mid \mathcal{B} \subseteq \mathcal{A} \text{ and } \mathcal{B} \text{ is finitely generated}\}.$$

If $\mathcal{A}$ is compact, then finitely many of those subvarieties suffice, and it follows that $\mathcal{A}$ is finitely generated. Conversely, assume that $\mathcal{A} = \mathrm{Var}(\mathbf{A})$, for some finite algebra $\mathbf{A}$. Let $\{\mathcal{V}_i \mid i \in I\}$ be an indexed collection of subvarieties of $\mathcal{V}$ and assume $\mathcal{A} \subseteq \bigvee_{i\in I}\mathcal{V}_i$. Then $\mathbf{A} \in \mathsf{HSP}(\bigcup_{i\in I}\mathcal{V}_i)$. Since $\mathbf{A}$ is finite, there is set of algebras $\mathcal{K} \subseteq \bigcup_{i\in I}\mathcal{V}_i$ and a finitely generated algebra $\mathbf{B} \in \mathsf{ISP}(\mathcal{K})$ such that $\mathbf{A}$ is a homomorphic image of $\mathbf{B}$. By local finiteness, $\mathbf{B}$ is finite, so there is a finite $\mathcal{K}' \subseteq \mathcal{K}$ such that $\mathbf{B} \in \mathsf{ISP}(\mathcal{K}')$. Then there is a finite $J \subseteq I$ such that $\mathcal{K}' \subseteq \bigcup_{j\in J}\mathcal{V}_j$, implying $\mathbf{A} \in \mathrm{Var}(\bigcup_{j\in J}\mathcal{V}_j) = \bigvee_{j\in J}\mathcal{V}_j$. So $\mathrm{Var}(\mathbf{A}) \subseteq \bigvee_{j\in J}\mathcal{V}_j$. Since $\mathcal{A} = \mathrm{Var}(\mathbf{A})$, it follows that $\mathcal{A}$ is compact. □

In the next result, by using the results just proved and some simple applications of Jónsson's Lemma, we give some new proofs of some known results. For example, part (3) was proved by Day [28, Corollary 3.8] using finitely projected algebras, and part (4) was proved by Davey [24, Theorem 3.3]. We are unaware of any reference to the first part, despite the simplicity of the observation. The proof is due to Brian Davey.

Corollary 3.3.4. *Let $\mathcal{V}$ be a locally finite congruence-distributive variety.*

(1) *$\mathcal{L}(\mathcal{V})$ is a distributive doubly algebraic lattice; hence it forms a double Heyting algebra.*

(2) *The following are equivalent for a variety $\mathcal{A} \in \mathcal{L}(\mathcal{V})$:*

   (i) *$\mathcal{A}$ is generated by a finite algebra and is join-irreducible in $\mathcal{L}(\mathcal{V})$;*

   (ii) *$\mathcal{A}$ is completely join-prime in $\mathcal{L}(\mathcal{V})$;*

   (iii) *$\mathcal{A} = \mathrm{Var}(\mathbf{A})$, for some unique (up to isomorphism) finite subdirectly irreducible algebra $\mathbf{A}$.*

(3) *Every finite subdirectly irreducible algebra in $\mathcal{V}$ is a splitting algebra.*

(4) *$\mathcal{L}(\mathcal{V})$ is isomorphic to $\mathcal{O}(\mathcal{V}_{\mathrm{fsi}})$, where $\mathcal{V}_{\mathrm{fsi}}$ is a set of representatives of the finite subdirectly irreducible algebras in $\mathcal{V}$, ordered by $\mathbf{A} \le \mathbf{B}$ if and only if $\mathbf{A} \in \mathsf{HS}(\mathbf{B})$.*

Proof. Part (1) follows from the previous lemma. Because $\mathcal{L}(\mathcal{V})$ is complete, part (3) follows from Lemma 3.2.2 and the implication (2iii) $\Rightarrow$ (2ii). Part (4) also follows from part (2). Firstly, by Theorem 3.3.1, we know that $\mathcal{L}(\mathcal{V}) \cong \mathcal{O}(P)$, where P is the ordered set of completely join-prime elements of $\mathcal{L}(\mathcal{V})$. By part (2), these are exactly the varieties generated by finite subdirectly irreducible algebras. Then by Jónsson's Lemma, we have $\mathrm{Var}(\mathbf{A}) \subseteq \mathrm{Var}(\mathbf{B})$ if and only if $\mathbf{A} \in \mathsf{HS}(\mathbf{B})$, so we see that P and $\mathcal{V}_{\mathrm{fsi}}$ are order-isomorphic.

We now prove part (2). Let $\mathcal{A} \in \mathcal{L}(\mathcal{V})$. First assume (2i). Because $\mathcal{L}(\mathcal{V})$ is distributive, it follows that $\mathcal{A}$ is join-prime. By Lemma 3.3.3, $\mathcal{A}$ is compact, which ensures that $\mathcal{A}$ is completely join-prime. Now assume (2ii). By Lemma 3.2.2 and Lemma 3.2.8, there exists a finitely generated subdirectly irreducible algebra $\mathbf{A}$ such that $\mathcal{A} = \mathrm{Var}(\mathbf{A})$. Because $\mathcal{V}$ is locally finite, $\mathbf{A}$ is finite. Uniqueness follows from Jónsson's Lemma. Indeed, if $\mathrm{Var}(\mathbf{B}) = \mathrm{Var}(\mathbf{A})$, then $\mathbf{B} \in \mathsf{HS}(\mathbf{A})$ and $\mathbf{A} \in \mathsf{HS}(\mathbf{B})$, so $\mathbf{A} \cong \mathbf{B}$. Now assume (2iii). To show that (2i) holds, by distributivity we need only prove that $\mathcal{A}$ is join-prime. Let $\mathbf{B}, \mathbf{C} \in \mathcal{V}$ and assume $\mathrm{Var}(\mathbf{A}) \subseteq \mathrm{Var}(\mathbf{B}) \vee \mathrm{Var}(\mathbf{C})$. Since $\mathbf{A}$ is subdirectly irreducible, by Jónsson's Lemma we have $\mathbf{A} \in \mathrm{Var}(\mathbf{B})$ or $\mathbf{A} \in \mathrm{Var}(\mathbf{C})$. Then $\mathrm{Var}(\mathbf{A}) \subseteq \mathrm{Var}(\mathbf{B})$ or $\mathrm{Var}(\mathbf{A}) \subseteq \mathrm{Var}(\mathbf{B})$, as required. □

For other examples of double Heyting algebras, an obvious place to look is within established classes of Heyting algebras.

3.4 Topoi and topology

Let $\langle X; \mathcal{T} \rangle$ be a topological space. It is easily verified that $\mathcal{T}$, ordered by set inclusion, is a complete lattice. Indeed, for any given $K \subseteq \mathcal{T}$,

$$\bigvee K = \bigcup K,$$
$$\bigwedge K = \mathrm{int}\left(\bigcap K\right),$$

where $\mathrm{int}(Y)$ is the *interior* of $Y \subseteq X$, i.e., the largest open set contained in Y. In particular, if K is finite, then $\bigwedge K = \bigcap K$. We will identify $\mathcal{T}$ with the lattice $\langle \mathcal{T}; \vee, \wedge \rangle$. It follows immediately that $\mathcal{T}$ is a join-infinite distributive lattice:

$$U \cap \bigcup K = \bigcup \{U \cap V \mid V \in K\},$$

for every $K \subseteq \mathcal{T}$. Thus, $\mathcal{T}$ forms a Heyting algebra by Example 3.1.2.

The natural follow-up question in the context of this thesis is, when does $\mathcal{T}$ form a double Heyting algebra? An obvious instance is an *Alexandrov topology*, wherein arbitrary intersections of open sets are open. This implies that $\bigwedge K$ as above is just $\bigcap K$, so the lattice is completely distributive. We claim that this example is not as interesting as it may seem at first glance. Firstly, if $\mathcal{T}$ is Alexandrov, then it is a complete lattice of sets, so it is isomorphic to a downset lattice (see Davey and Priestley [26, Theorem 10.29]). Conversely, any downset lattice is closed under arbitrary unions and intersections, so it defines an Alexandrov topology. Hence, we have only revisited Example 3.1.2 with less generality.

One may consider other conditions imposed on topological spaces instead. Two distinct points $x, y \in X$ are *topologically distinguishable* if there is an open set containing one but not the other. Then X is called a *T_0 space* if every pair of distinct points are topologically distinguishable. It is a *T_1 space* if the set $\{x\}$ is closed, for every $x \in X$. Let us say that a topological space is *dually pseudocomplemented* if the lattice of open sets forms an H^+-algebra. Fellow graduate student Tim Koussas provided the next observation.

Proposition 3.4.1. *Let $\langle X; \mathcal{T}\rangle$ be a T_1 space. If $\sim U$ exists in $\mathcal{T}$, then $\sim U = X \backslash U$. Thus, $\sim U$ exists in $\mathcal{T}$ if and only if U is clopen. It follows that a T_1 space is dually pseudocomplemented if and only if it is discrete.*

Proof. Let $U \in \mathcal{T}$ and assume $\sim U$ exists in $\mathcal{T}$. Suppose there exists $x \in U \cap \sim U$. Then by assumption, $\sim U \backslash \{x\}$ is open. But $\sim U \backslash \{x\} \cup U = X$, which contradicts the minimality of $\sim U$. Hence $\sim U = X \backslash U$. For the last statement, the set $U := X \backslash \{x\}$ is open, for every $x \in X$. So if X is dually pseudocomplemented, then $\sim U = \{x\}$ is an open set. Then since singleton sets are open, the topology is discrete. □

Let $\langle X; \mathcal{T}\rangle$ be a topological space. Then the quotient of X obtained by identifying indistinguishable points is a T_0 space. Denote the lattice of open sets of that quotient by $\mathcal{T}_0$. It is easy to show that the quotient map induces a lattice isomorphism from $\mathcal{T}$ to $\mathcal{T}_0$. The proof is not difficult, but we omit it here. The isomorphism means that, as far as this section is concerned, we may assume that X is a T_0 space. From the previous proposition, a nice place to look for a dually pseudocomplemented topological space would be somewhere in the class of T_0 spaces that are neither T_1 nor Alexandrov. Many everyday topological spaces are T_1, such as Euclidean spaces, so examples of dually pseudocomplemented topological spaces that are not Alexandrov may be quite elusive, should they even exist at all.

Open Problem 4. Let $\langle X; \mathcal{T}\rangle$ be a topological space and assume that X is T_0 but not T_1. When does $\mathcal{T}$ form a double Heyting algebra? More generally, when is $\mathcal{T}$ dually pseudocomplemented? Is it a topologically meaningful assumption? It would also be interesting to find a topological space that is dually pseudocomplemented but does not form a double Heyting algebra.

A related concept in category theory, inspired by properties of topological spaces, is that of a *topos*. We reserve the formalities of what follows to another thesis—the details are beyond our current scope. For the categorically inclined reader, a topos is a category $\mathbf{C}$ with three core components:

(1) All finite limits and colimits exist in $\mathbf{C}$.

(2) All exponentials exist in $\mathbf{C}$.

(3) $\mathbf{C}$ has a subobject classifier.

Note that the category of topological spaces with continuous maps is not a topos, since it does not have all exponentials. Let $\mathbf{C}$ be a category and let X be an object in $\mathbf{C}$. A *subobject* of X is a pair consisting of an object Y and a monomorphism from Y to X. Informally, a subobject classifier is then an object Ω in $\mathbf{C}$ such that every subobject of X is determined by a morphism from X to Ω and vice versa. In the case of sets, a subobject is just a set with a one-to-one map, and the subobject classifier is the set $\{0, 1\}$. Denote the class of isomorphism classes of subobjects of X by $\mathrm{Sub}(X)$. If $\mathrm{Sub}(X)$ is a set, then there is an appropriate order on $\mathrm{Sub}(X)$ that generalises the subset relation for sets. Moreover, if $\mathbf{C}$ is a topos and $\mathrm{Sub}(X)$ is a set, then $\mathrm{Sub}(X)$ forms a Heyting algebra.

Lawvere showed in [66] that if a category $\mathbf{C}$ is a "presheaf topos", then subobject sets in $\mathbf{C}$ also form double Heyting algebras. Reyes and Zolfaghari [78] extended this by characterising the toposes such that every subobject set is a double Heyting algebra. To avoid dwelling on the underlying category theory, we will not include it here. One important example of a presheaf topos given by Reyes and Zolfaghari is the category of directed multigraphs. The morphisms must be defined carefully as maps on vertices *and* edges. Then the subobject classifier consists of a pair of vertices with no edges except for a pair of loops on one of the vertices. The exponential for directed graphs will be discussed in Section 3.6. Notably, it follows that the lattice of subgraphs of a graph forms a double Heyting algebra. We will now investigate this further.

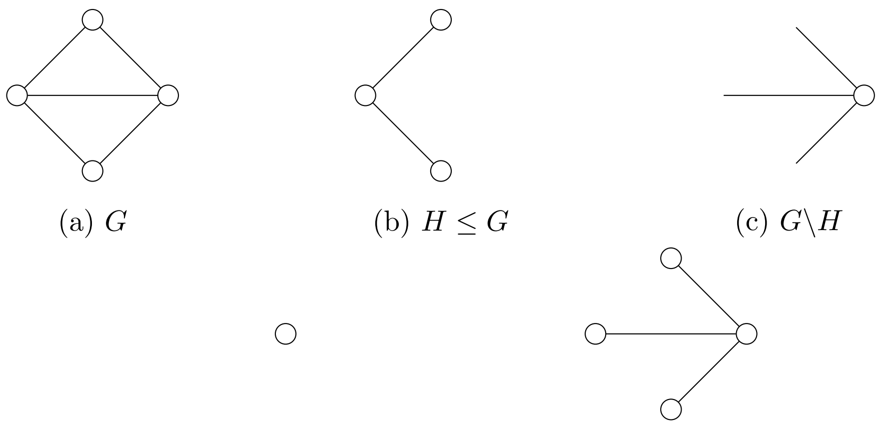
3.5 Lattices of subgraphs

Definition 3.5.1. A *directed graph* (digraph for short) is a pair $\langle V, E\rangle$ such that E is a binary relation on V. We say that V is the set of *vertices* and E is the set of *edges*. If G is a directed graph, then let V_G denote its vertices and let E_G denote its edges. If E_G is a symmetric relation, then G is a *graph* (which permits loops). A digraph H is a *subgraph* of G if $V_H \subseteq V_G$ and $E_H \subseteq E_G$. Let $\mathcal{S}(G)$ denote the set of subgraphs of G.

As we just noted, from Reyes and Zolfaghari's observation, the set of subgraphs of a graph forms a double Heyting algebra. The categorical background is unnecessary to prove this. It is easy to see that $\mathcal{S}(G)$ is a lattice. Indeed, if $K \subseteq \mathcal{S}(G)$, then

$$\bigvee K = \left\langle \bigcup\{V_H \mid H \in K\}, \bigcup\{E_H \mid H \in K\} \right\rangle,$$
$$\bigwedge K = \left\langle \bigcap\{V_H \mid H \in K\}, \bigcap\{E_H \mid H \in K\} \right\rangle.$$

Consequently, the set of subgraphs of a digraph is a completely distributive complete lattice, which forms a double Heyting algebra by Example 3.1.2. To elucidate the structure of these lattices, let G be the graph in Figure 3.3a and let H be the subgraph of G given in Figure 3.3b. If we try to take the set complement of H in G, we are left with what is shown in Figure 3.3c. Clearly, what remains is not a graph. There are two immediately obvious choices to fix the complement. We can abandon the superfluous edges, resulting in the graph in Figure 3.3d. Alternatively, we could reintroduce the required missing vertices as in Figure 3.3e. The debate over which is the correct complement has been going on for centuries⸮[2]



(a) G (b) $H \leq G$ (c) $G \backslash H$

(d) $\neg H$ (e) $\sim H$

Figure 3.3

The debate is resolved by observing that the two candidates for complementation correspond respectively to the pseudocomplement and the dual pseudocomplement

[2]The punctuation mark ⸮, known as a *percontation point*, is a non-standard symbol used to denote sarcasm or irony.

in $\mathcal{S}(G)$. We will not describe the relative pseudocomplement and dual relative pseudocomplement operations, for reasons that will be clear in the next chapter. The obvious question to ask now is, which double Heyting algebras are isomorphic to a lattice of subgraphs? We will resolve this question in Chapter 4.

3.6 The graph homomorphism lattice

Another example of a double Heyting algebra, coincidentally also connected to graph theory, is the digraph homomorphism lattice. This lattice is a well-known Heyting algebra. The fact that it forms a dual Heyting algebra is apparently unknown. This was observed and proved by Brian Davey, and his proof is a simple application of Lemma 3.1.8 which we will present here.

Definition 3.6.1. Let G and H be finite digraphs. A map $\varphi\colon V_G \to V_H$ is a *digraph homomorphism* (homomorphism for short) provided that $(v_1, v_2) \in E_G$ implies $(\varphi(v_1), \varphi(v_2)) \in E_H$. We will unambiguously write $\varphi\colon G \to H$ rather than $\varphi\colon V_G \to V_H$ and say that φ is a *homomorphism from G to H*. If there exists a homomorphism $\varphi\colon G \to H$, then we write $G \mapsto H$. If $G \mapsto H$ and $H \mapsto G$, then G and H are *homomorphically equivalent*. The class of all finite digraphs homomorphically equivalent to G will be denoted by $[G]$.

Remark 3.6.2. Note that these are not the same as the morphisms in the category of directed multigraphs considered earlier.

Let $\mathcal{G}$ be the set of finite digraphs (*viz.* a set of representatives for the class of finite digraphs). It is easily seen that the relation $\mapsto$ on $\mathcal{G}$ is a quasi-order, but it is not a partial order because it is not antisymmetric. By identifying the equivalence classes, we obtain a partial order defined by $[G] \leq [H]$ if and only if $G \mapsto H$. This order is known as *the digraph homomorphism order*. To study the structure, it is helpful to have a standard representative for each class, which we will now describe.

Definition 3.6.3. Let G be a finite digraph and let H be a subgraph of G. A *retration* of G to H is a homomorphism $\varphi\colon G \to H$ such that $\varphi(v) = v$, for all $v \in V_H$. A *core* is a finite digraph that has no retraction to any of its subgraphs. If H is a core and $H \in [G]$, then we say H is a *core of G*.

Proposition 3.6.4 (Hell and Nešetřil [43, Corollary 1.32]). *Let G be a finite digraph. Up to isomorphism, there is only one core in $[G]$.*

Thus, the digraph homomorphism order can be seen as an ordering on cores instead.

Definition 3.6.5. Let $\mathcal{C}$ be a set of representatives of the class of cores. That is, for every core G, there exists exactly one G' in $\mathcal{C}$ such that G and G' are isomorphic. We will identify $\mathcal{C}$ with the ordered set $\langle \mathcal{C}; \leq \rangle$ defined by $G_1 \leq G_2$ if and only if $G_1 \mapsto G_2$. For a finite digraph G, let $[G]_{\mathcal{C}}$ denote its core in $\mathcal{C}$.

From now on, $\mathcal{C}$ will be a fixed set of representatives as above. To describe the lattice structure, we require two graph constructions.

Definition 3.6.6. Let G and H be digraphs. The disjoint union of G and H is denoted by $G \mathbin{\dot\cup} H$. The (direct) product of G and H, denoted $G \times H$, is given by

$$\begin{aligned} V_{G\times H} &= V_G \times V_H, \\ E_{G\times H} &= \big\{\big((a,x),(b,y)\big) \mid (a,b) \in E_G \text{ and } (x,y) \in E_H\big\}. \end{aligned}$$

Notice that this is not the graph-theoretic Cartesian product.

Proposition 3.6.7 (Hell and Nešetřil [43, Proposition 3.2])**.** *The ordered set $\mathcal{C}$ is a bounded distributive lattice. Specifically, for all $G, H \in \mathcal{C}$,*

$$\begin{aligned} G \vee H &= [G \mathbin{\dot\cup} H]_{\mathcal{C}}, \\ G \wedge H &= [G \times H]_{\mathcal{C}}. \end{aligned}$$

The bottom is a graph consisting of a single vertex and no edges, and the top is a graph consisting of a single vertex with a loop. Moreover, the bottom is completely meet-irreducible, and its cover is a graph with two vertices connected by one directed edge.

Thus, we will call $\mathcal{C}$ *the digraph homomorphism lattice*. For the Heyting implication, the exponential graph is required.

Definition 3.6.8. Let G and H be finite digraphs. Define the *exponential digraph* G^H as follows.

(1) The set of vertices of G^H is the set of all functions $f\colon V_G \to V_H$.

(2) A pair (f_1, f_2) is an edge in G^H if and only if $(f_1(v_1), f_2(v_2)) \in E_H$, for all $(v_1, v_2) \in E_G$.

Proposition 3.6.9 (Hell and Nešetřil [43, Proposition 2.18])**.** *If F, G, and H are finite digraphs, then $G \times F \mapsto H$ if and only if $F \mapsto H^G$.*

Corollary 3.6.10. *The lattice $\mathcal{C}$ forms a Heyting algebra, wherein $H \to G = [G^H]_{\mathcal{C}}$.*

Recall that a graph is a symmetric digraph, which permits loops. The disjoint union, the product, and the exponential of two graphs always produces another graph. Moreover, if G is a graph, then $[G]_{\mathcal{C}}$ is a graph. Thus, the subset of $\mathcal{C}$ given by those elements that are graphs is a Heyting subalgebra of $\mathcal{C}$.

Definition 3.6.11. Let $\mathcal{C}_S$ denote the Heyting subalgebra of $\mathcal{C}$ consisting of the graphs in $\mathcal{C}$, and call it *the graph homomorphism lattice.*

Note that if G is a graph, then $[G]_{\mathcal{C}} \in \mathcal{C}_S$, so the notation $[G]_{\mathcal{C}}$ is unambiguous. The following theorem illustrates the remarkable complexity of both the graph and digraph homomorphism lattices.

Theorem 3.6.12 (Hubička and Nešetřil [48])**.** *Every countable ordered set order-embeds into $\mathcal{C}_S$, and therefore into $\mathcal{C}$ as well.*

The proof that the two lattices form double Heyting algebras relies on the following simple result.

Lemma 3.6.13. *A core G is join-irreducible in $\mathcal{C}$ (resp. $\mathcal{C}_S$) if and only if there is a connected digraph (resp. graph) H such that $[H]_{\mathcal{C}} = G$.*

Proof. Let H be a connected digraph and let $G = [H]_{\mathcal{C}}$. Assume that $[X \dot\cup Y]_{\mathcal{C}} = G$. Then $H \mapsto X \dot\cup Y$, but since H is connected, its image must be connected. Hence, by restricting the codomain, there is also a homomorphism from H to X or from H to Y. If $H \mapsto X$, then $[H]_{\mathcal{C}} \leq [X]_{\mathcal{C}}$, and we have $[X]_{\mathcal{C}} \leq [H]_{\mathcal{C}}$ by assumption. So in this case $[H]_{\mathcal{C}} = [X]_{\mathcal{C}}$. The same argument applies with Y instead of X, so G is join-irreducible. For the converse, we first have that G consists of some number of connected components, say $G_1, \ldots, G_n$. Then $G = [G_1 \dot\cup \ldots \dot\cup G_n]_{\mathcal{C}}$, and if G is join-irreducible, it follows that $G = [G_i]_{\mathcal{C}}$, for some $i \leq n$. □

Corollary 3.6.14. *Every element of $\mathcal{C}$ (resp. $\mathcal{C}_S$) is the join of finitely many join-irreducible elements from $\mathcal{C}$ (resp. $\mathcal{C}_S$).*

In a distributive lattice, an element is join-prime if and only if it is join-irreducible, so by Lemma 3.1.8, both $\mathcal{C}$ and $\mathcal{C}_S$ are double Heyting algebras. One can also see from the proof of Lemma 3.1.8 that if G and H are graphs, then $[G]_{\mathcal{C}} \dot- [H]_{\mathcal{C}}$ is also a graph.

Corollary 3.6.15. *Both $\mathcal{C}$ and $\mathcal{C}_S$ form double Heyting algebras. Moreover, $\mathcal{C}_S$ is a double Heyting subalgebra of $\mathcal{C}$.*

Open Problem 5. What is the equational theory of the double Heyting algebras $\mathcal{C}$ and $\mathcal{C}_S$? Interestingly, the two lattices are both simple double Heyting algebras. This is because the bottom of $\mathcal{C}$ is completely meet-irreducible, so $\neg{\sim}[G]_{\mathcal{C}} = 0$, for all $[G]_{\mathcal{C}} \neq 1$. With that in mind, give a subdirect product representation of the double Heyting algebras obtained by truncating $\mathcal{C}$ and $\mathcal{C}_S$ at the bottom.

Open Problem 6. The direct product of digraphs defined in this thesis is also known as the *categorical product*. True to the name, it is the categorical product with respect to digraph homomorphisms. Various other products are studied in graph theory. There is an enormous literature on these products, described extensively by Hammack, Imrich, and Klavžar [39]. The so-called "four standard graph products" are the direct product, the Cartesian product, the strong product, and the lexicographic product (see [39]). The exponential graph is the Heyting implication, or in other words, it is residual of the direct product in $\mathcal{C}$. It has been rumoured that the Cartesian product is also residuated in $\mathcal{C}$. We have not yet investigated this, and we have heard no rumours on whether the strong product and lexicographic product are also residuated.

4

Lattices of incidence structures

We saw in Section 3.5 that the lattice of subgraphs of a graph forms a double Heyting algebra, where the pseudocomplements and dual pseudocomplements have a very natural interpretation. This raises to an obvious question: which double Heyting algebras are isomorphic to a lattice of subgraphs? We will see that the lattice of subgraphs actually forms a congruence-regular double p-algebra, which is then term-equivalent to a double Heyting algebra by Theorem 1.6.4. It is easy to create an example of a regular double p-algebra that is not isomorphic to a lattice of subgraphs. The source of the problem is twofold: two vertices can only have one edge between them, and that edge can connect at most two vertices. Allowing hyperedges and permitting multiple edges lets us use incidence structures, which are a standard generalisation of graphs. The lattice of point-preserving substructures of an incidence structure (see Definition 4.1.2) forms a regular double p-algebra, which includes the lattice of subgraphs of a graph as a special case. This includes powerset lattices as well, because the subgraphs of a graph with no edges whatsoever are just subsets of the vertices.

There are two main results to this chapter, split into several parts. Using the dual spaces of regular double p-algebras, we prove that every regular double p-algebra can be embedded into the lattice of point-preserving substructures of some incidence structure. It will follow that every finite regular double p-algebra is isomorphic to the substructure lattice of some incidence structure. For the second main result, we characterise substructure lattices. The characterisation is similar to the characterisation of powerset lattices as complete and atomic Boolean algebras. The results of this chapter have been published in *Algebra Universalis* [84]. Compared to that article, for this chapter we have made some statements more precise, improved the clarity of some proofs with extra details, updated the notation to use $\neg$ and $\sim$ instead of * and $^+$, and included examples to illustrate the constructions involved. Unless otherwise specified, all results in this chapter are the original work of the author.

4.1 Incidence structures

Definition 4.1.1. An *incidence structure* is a triple $G = \langle P, L, I \rangle$ such that I is a subset of $P \times L$. The set P is the set of *points* of G, the set L is the set of *lines*, and the relation I is the *incidence relation*. If $(p, l) \in I$, then we say that p *is incident to* l and that p *is a point of* l. A point that is not incident to any lines is called an *isolated point*. Similarly, if a line l has no points, then we say l is an *empty line*. We will permit the empty incidence structure $\langle \varnothing, \varnothing, \varnothing \rangle$.

An incidence structure is a bare-bones geometric object with very little actual structure, and provides a common generalisation of graphs and planes. We require no background knowledge of incidence geometry for this chapter, although the interested reader may consult De Bruyn [29] for more. Also worthy of note is that the definition of an incidence structure is identical to the definition of a context in formal concept analysis—see [35] for example.

Definition 4.1.2. Let $\langle P, L, I \rangle$ and $\langle P', L', I' \rangle$ be incidence structures. We will say that $\langle P', L', I' \rangle$ is a *point-preserving substructure of* $\langle P, L, I \rangle$ if all of the following hold:

(1) $P' \subseteq P$ and $L' \subseteq L$,

(2) $I' = I \cap (P' \times L')$,

(3) for all $l \in L'$, if $(p, l) \in I$, then $p \in P'$.

We will refer to an incidence structure satisfying (1) and (2) as a *substructure*, and condition (3) will be referred to by the phrase *point-preserving*. Let $\mathcal{S}(G)$ denote the ordered set of point-preserving substructures of G, where the order is defined by $H_1 \leq H_2$ if and only if H_1 is a substructure of H_2. To ease notation, if $H = \langle P', L', I' \rangle \in \mathcal{S}(G)$, then we will leave the incidence relation implicit and write $H = \langle P', L' \rangle$. We say that $H \in \mathcal{S}(G)$ is *proper* if $H \notin \{\langle \varnothing, \varnothing \rangle, G\}$.

Let $G = \langle P, L, I \rangle$ be an incidence structure. It is easy to see that $\mathcal{S}(G)$ is a complete distributive lattice, where $\langle \varnothing, \varnothing \rangle$ is the bottom and G is the top, and if $\{\langle P_j, L_j \rangle \mid j \in J\}$ is a family of point-preserving substructures of G, then joins and meets are given respectively by

$$\bigvee \{\langle P_j, L_j \rangle \mid j \in J\} = \left\langle \bigcup \{P_j \mid j \in J\}, \bigcup \{L_j \mid j \in J\} \right\rangle,$$
$$\bigwedge \{\langle P_j, L_j \rangle \mid j \in J\} = \left\langle \bigcap \{P_j \mid j \in J\}, \bigcap \{L_j \mid j \in J\} \right\rangle.$$

Let $H = \langle P_H, L_H \rangle$ be a point-preserving substructure of G. Define the operations ${\neg}H$ and ${\sim}H$ by

$$\neg H = \langle P \backslash P_H, \{l \in L \backslash L_H \mid (p,l) \in I \implies p \in P \backslash P_H\} \rangle,$$
$$\sim H = \langle (P \backslash P_H) \cup \{p \in P \mid (\exists l \in L \backslash L_H)\ (p,l) \in I\}, L \backslash L_H \rangle.$$

Less formally, ${\neg}H$ is obtained by taking the complement of the substructure and disposing of all lines with missing points. Similarly, ${\sim}H$ is obtained by taking the complement and then putting the missing points back. Observe that if G has an empty set of lines, then $\mathcal{S}(G)$ is isomorphic to the powerset of P, which we denote by $\mathcal{P}(P)$, and hence it is a Boolean lattice. Similarly, if G has no points, then $\mathcal{S}(G)$ is isomorphic to $\mathcal{P}(L)$.

Proposition 4.1.3. *Let $G = \langle P, L, I \rangle$ and let $H = \langle P_H, L_H \rangle$ be a point-preserving substructure of G. Then ${\neg}H$ and ${\sim}H$, as defined above, are the pseudocomplement and dual pseudocomplement of H in $\mathcal{S}(G)$, respectively.*

Proof. For convenience, write ${\neg}H = \langle P_{\neg}, L_{\neg} \rangle$ and ${\sim}H = \langle P_{\sim}, L_{\sim} \rangle$. It is clear from the definition that ${\neg}H$ and ${\sim}H$ are point-preserving substructures of G. It is also readily verified that $H \wedge {\neg}H = \langle \varnothing, \varnothing \rangle$ and $H \vee {\sim}H = \langle P, L, I \rangle$. Now let $K = \langle P_K, L_K \rangle \in \mathcal{S}(G)$.

To see that ${\neg}H$ is the pseudocomplement, assume that $K \wedge H = \langle \varnothing, \varnothing \rangle$. We show that $K \leq {\neg}H$. By assumption, $P_K \cap P_H = \varnothing$ and thus $P_K \subseteq P \backslash P_H = P_{\neg}$. We also have $L_K \cap L_H = \varnothing$, so $L_K \subseteq L \backslash L_H$. Let $l \in L_K$. Since K is point-preserving, if $(x,l) \in I$, then $x \in P_K \subseteq P \backslash P_H$, and so we have $l \in L_{\neg}$ by definition. Hence $L_K \subseteq L_{\neg}$, and therefore $K \leq {\neg}H$, as claimed.

Next, assume $K \vee H = G$. We must show that ${\sim}H \leq K$. We have $L_K \cup L_H = L$, and so $L_{\sim} = L \backslash L_H \subseteq L_K$. Now let $p \in P_{\sim}$. If $p \in P \backslash P_H$, then since $P_K \cup P_H = P$, we have $p \in P_K$. Otherwise, $p \in \{p \in P \mid (\exists l \in L \backslash L_H)\ (p,l) \in I\}$. In that case, there exists $l \in L \backslash L_H$ such that $(p,l) \in I$. Since $L \backslash L_H \subseteq L_K$, it follows that $l \in L_K$. Since K is point-preserving, we then must have $p \in P_K$, and hence $P_{\sim} \subseteq P_K$. We conclude that ${\sim}H \leq K$, as required. □

Therefore, $\mathcal{S}(G)$ forms a distributive double p-algebra. Henceforth, we will identify the algebra $\langle \mathcal{S}(G); \vee, \wedge, \neg, \sim, \langle \varnothing, \varnothing \rangle, G \rangle$ with its underlying set $\mathcal{S}(G)$. Recall from Theorem 1.6.2 that a double p-algebra is regular if and only if it satisfies the following implication:

$$\text{if } \neg x = \neg y \text{ and } {\sim}x = {\sim}y, \text{ then } x = y.$$

Definition 4.1.4. Let $\mathbf{L}$ be a lattice. We say that $a \in L$ is an *atom* if $0 \prec a$. Dually, we say that $c \in L$ is a *coatom* if $c \prec 1$. We let $\mathcal{A}(\mathbf{L})$ denote the set of atoms in $\mathbf{L}$ and $\mathcal{C}(\mathbf{L})$ denote the set of coatoms in $\mathbf{L}$. For all $x \in L$, we denote by $\mathcal{A}_{\mathbf{L}}(x)$ the set of atoms below x, that is, $\mathcal{A}_{\mathbf{L}}(x) = {\downarrow}x \cap \mathcal{A}(\mathbf{L})$. Similarly, let $\mathcal{C}_{\mathbf{L}}(x) = {\uparrow}x \cap \mathcal{C}(\mathbf{L})$. We will typically omit the subscript $\mathbf{L}$, unless the distinction is required. We say that $\mathbf{L}$ is *atomic* if $\mathcal{A}(x)$ is non-empty, for every $x \in L\backslash\{0\}$. Similarly, $\mathbf{L}$ is *coatomic* if $\mathcal{C}(x)$ is non-empty, for all $x \in L\backslash\{1\}$. If $\mathbf{L}$ is both atomic and coatomic, we say that $\mathbf{L}$ is *doubly atomic.*

Theorem 4.1.5. *Let $G = \langle P, L, I\rangle$ be an incidence structure. Then $\mathcal{S}(G)$ is a completely distributive doubly atomic regular double p-algebra.*

Proof. The fact that $\mathcal{S}(G)$ is completely distributive follows readily from the distributivity of set union and intersection. There are two types of atoms in $\mathcal{S}(G)$:

(1) for each point p, the substructure $\langle\{p\}, \varnothing\rangle$,

(2) for each empty line l, the substructure $\langle\varnothing, \{l\}\rangle$.

If a non-empty point-preserving substructure of G contains at least one point, then it lies above an atom of the first type. Otherwise, it only consists of empty lines, so it must be above an atom of the second type. Hence $\mathcal{S}(G)$ is atomic. Similarly, there are two types of coatoms:

(1) for each line l, the substructure $\langle P, L\backslash\{l\}\rangle$,

(2) for each isolated point p, the substructure $\langle P\backslash\{p\}, L\rangle$.

If a proper point-preserving substructure of G is missing a line, then it is below a coatom of the first type. Otherwise, it contains all lines and is missing at least one point, which must be isolated because the substructure is point-preserving. So it is below a coatom of the second type. It follows that $\mathcal{S}(G)$ is coatomic.

Lastly, for regularity, let $H = \langle P_H, L_H\rangle$ and $K = \langle P_K, L_K\rangle$ be point-preserving substructures of G. If $\neg H = \neg K$, then $P\backslash P_H = P\backslash P_K$, and if ${\sim}H = {\sim}K$, then $L\backslash L_H = L\backslash L_K$. It follows that if both $\neg H = \neg K$ and ${\sim}H = {\sim}K$, then $P_H = P_K$ and $L_H = L_K$. Therefore, $\mathcal{S}(G)$ is regular. □

Lemma 4.1.6. *Let $\{G_j \mid j \in J\}$ be a set of incidence structures that have pairwise disjoint sets of points and pairwise disjoint sets of lines. Then*

$$\mathcal{S}\Big(\bigcup_{j\in J} G_j\Big) \cong \prod_{j\in J} \mathcal{S}(G_j).$$

Proof. It is easily verified that the map $\varphi\colon \mathcal{S}(\bigcup_{j\in J} G_j) \to \prod_{j\in J} \mathcal{S}(G_j)$ defined by $\varphi(H)(j) = H \wedge G_j$ is the required isomorphism. □

In summary, the lattice of point-preserving substructures of an incidence structure forms a completely distributive doubly atomic regular double p-algebra, and products of the lattices correspond to disjoint unions of the structures.

Open Problem 7. In terms of the incidence structure, how can we describe subalgebras and homomorphic images of the double p-algebra $\mathcal{S}(G)$?

4.2 Properties of double p-algebras

Some preliminary results are required to prove the representation. The proof of the next lemma is straightforward.

Lemma 4.2.1. *Let* $\mathbf{A}$ *be a doubly atomic distributive lattice, let* $x, y \in A$*, and let* $X \subseteq A$*. Then, whenever the relevant meets and joins exist in* $\mathbf{A}$:

(1) $\mathcal{A}(\bigwedge X) = \bigcap\{\mathcal{A}(x) \mid x \in X\}$,

(2) $\mathcal{C}(\bigvee X) = \bigcap\{\mathcal{C}(x) \mid x \in X\}$,

(3) $\mathcal{A}(x \vee y) = \mathcal{A}(x) \cup \mathcal{A}(y)$,

(4) $\mathcal{C}(x \wedge y) = \mathcal{C}(x) \cup \mathcal{C}(y)$,

(5) *if* $\mathbf{A}$ *is complete and satisfies* (JID)*, then* $\mathcal{A}(\bigvee X) = \bigcup\{\mathcal{A}(x) \mid x \in X\}$,

(6) *if* $\mathbf{A}$ *is complete and satisfies* (MID)*, then* $\mathcal{C}(\bigwedge X) = \bigcup\{\mathcal{C}(x) \mid x \in X\}$.

The next lemma is a trivial but critical observation: in a complete doubly atomic regular double p-algebra, each element is determined precisely by the atoms below and coatoms above it.

Lemma 4.2.2. *Let* $\mathbf{A}$ *be a complete double p-algebra and let* $x, y \in A$.

(1) *If* $\mathbf{A}$ *is atomic, then* $\neg x = \neg\bigvee\mathcal{A}(x)$.

(2) *If* $\mathbf{A}$ *is coatomic, then* $\sim x = \sim\bigwedge\mathcal{C}(x)$.

(3) *If* $\mathbf{A}$ *is doubly atomic and regular, then* $\mathcal{A}(x) = \mathcal{A}(y)$ *and* $\mathcal{C}(x) = \mathcal{C}(y)$ *together imply* $x = y$.

Proof. Part (3) follows from (1) and (2). For part (1), we have $\bigvee\mathcal{A}(x) \leq x$, and since $\neg$ is order-reversing, we have $\neg\bigvee\mathcal{A}(x) \geq \neg x$. For the converse, suppose that $\neg\bigvee\mathcal{A}(x) \wedge x \neq 0$. Then there is some $a \in \mathcal{A}(x)$ such that $a \leq \neg\bigvee\mathcal{A}(x)$. But $a \leq \bigvee\mathcal{A}(x)$, so $a \leq \bigvee\mathcal{A}(x) \wedge \neg\bigvee\mathcal{A}(x) = 0$, a contradiction. Thus, $\neg\bigvee\mathcal{A}(x) \wedge x = 0$, so that $\neg\bigvee\mathcal{A}(x) \leq \neg x$, as needed. A dual argument holds for (2). □

Definition 4.2.3. Let $\mathbf{A}$ be a lattice. For all $a \in A$, we let $h_a \colon A \to {\downarrow}a$ denote the map given by $x \mapsto x \wedge a$. If $\mathbf{A}$ is a double p-algebra, we further let $\neg_a x = \neg x \wedge a$ and let $\sim_a x = {\sim}x \wedge a$.

Recall that the center of a distributive lattice $\mathbf{A}$, denoted by $\mathrm{Cen}(\mathbf{A})$, is the set of complemented elements of $\mathbf{A}$.

Lemma 4.2.4. *Let $\mathbf{A}$ be a distributive double p-algebra. Let $a \in \mathrm{Cen}(\mathbf{A})$ and let $x \in A$. Then $\langle {\downarrow}a; \vee, \wedge, \neg_a, \sim_a, 0, a\rangle$ is a distributive double p-algebra and the map $h_a \colon A \to {\downarrow}a$ is a double p-algebra homomorphism.*

Proof. Let $\mathbf{B} = \langle {\downarrow}a; \vee, \wedge, \neg_a, \sim_a, 0, a\rangle$, so that h_a maps $\mathbf{A}$ onto $\mathbf{B}$. Distributivity guarantees that h_a preserves $\vee$. We have

$$\neg_a h_a(x) = \neg h_a(x) \wedge a = \neg(x \wedge a) \wedge a = \neg x \wedge a = h_a(\neg x),$$

and because $a \in \mathrm{Cen}(\mathbf{A})$, we have ${\sim}(x \wedge a) = {\sim}x \vee {\sim}a = {\sim}x \vee \neg a$, so

$$\sim_a h_a(x) = {\sim}(x \wedge a) \wedge a = ({\sim}x \vee \neg a) \wedge a = {\sim}x \wedge a = h_a({\sim}x).$$

It is obvious that the remaining operations are preserved by h_a. Hence $h_a \colon \mathbf{A} \to \mathbf{B}$ is a surjective homomorphism, and it follows that $\mathbf{B}$ is a distributive double p-algebra. □

It is well known that if $\mathbf{L}$ is a distributive lattice and $a \in \mathrm{Cen}(\mathbf{A})$, then $\mathbf{L}$ is isomorphic to $h_a(\mathbf{L}) \times h_{\neg a}(\mathbf{L})$. A map defined on a double p-algebra is a double p-algebra isomorphism if and only if it is an order-isomorphism. The next lemma is then immediate.

Lemma 4.2.5. *Let $\mathbf{A}$ be a double p-algebra and let $a \in \mathrm{Cen}(\mathbf{A})$. Then $\mathbf{A}$ is isomorphic to $h_a(\mathbf{A}) \times h_{\neg a}(\mathbf{A})$.*

4.3 Embedding substructure lattices

We note that this section can be read independently of the next: none of the results here are used to prove the main result of Section 4.4. To show that not every regular double p-algebra is isomorphic to a substructure lattice, we can use any Boolean counterexample, e.g., the finite and cofinite subsets of the natural numbers. Similarly, just as every Boolean algebra embeds into a powerset lattice, we prove here that every regular double p-algebra embeds into a substructure lattice. The incidence structure is determined entirely by the ordered set of prime filters.

Definition 4.3.1. Let X be an ordered set and assume every element of X is minimal or maximal. Then let $\mathcal{E}(X)$ be the incidence structure $\langle P, L, I\rangle$ defined by

$$P = \max(X),$$
$$L = X \backslash \max(X),$$
$$I = \{(x, y) \in P \times L \mid x > y\}.$$

This just says that the non-maximal elements are the lines and the points incident to a given line are the maximal elements above it. Note that an element that is both minimal and maximal is treated as an isolated point. Theorem 1.6.2 tells us that every prime filter in a regular double p-algebra is minimal or maximal, so if $\mathbf{A}$ is a regular double p-algebra and $X = \mathcal{F}_p(\mathbf{A})$, then $P \cup L = \mathcal{F}_p(\mathbf{A})$.

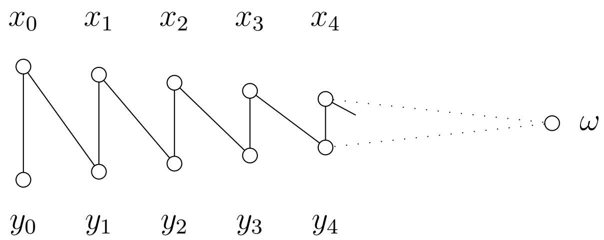


Figure 4.1

To illustrate the construction, consider the ordered set shown in Figure 4.1. Then $P = \{\omega\} \cup \{x_i \mid i \in \omega\}$ and $L = \{y_i \mid i \in \omega\}$. The only element above y_0 is x_0, so the line y_0 is incident only to the point x_0. In other words, y_0 is a loop on x_0. Every other line y_{i+1} is incident only to the points x_i and x_{i+1}. The point ω is an isolated point. The incidence structure is an infinite graph, and is illustrated in Figure 4.2.

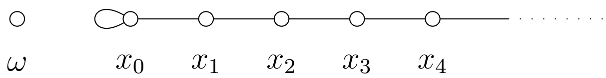


Figure 4.2

Theorem 4.3.2. *Let X be an ordered set and assume every element of X is minimal or maximal. Then $\mathcal{S}(\mathcal{E}(X)) \cong \mathcal{U}(X)$. Consequently, if $\mathbf{A}$ is a regular double p-algebra, then $\mathbf{A}$ embeds into $\mathcal{S}(\mathcal{E}(\mathcal{F}_p(\mathbf{A})))$.*

Proof. First observe that $\mathcal{U}(X)$ is a double p-algebra where, for all $U \in \mathcal{U}(X)$, we have $\neg U = X \backslash{\downarrow} U$ and $\sim U = {\uparrow}(X \backslash U)$. To see that the second claim follows from the first, substitute $\mathcal{F}_p(\mathbf{A})$ for X. Since the double p-algebra $\mathcal{U}^{\mathcal{T}}(\mathcal{F}_p(\mathbf{A}))$ is a subalgebra of $\mathcal{U}(\mathcal{F}_p(\mathbf{A}))$, we then have $\mathbf{A} \leq \mathcal{S}(\mathcal{E}(\mathcal{F}_p(\mathbf{A})))$.

Now let $G = \mathcal{E}(X)$ and define the map $\varphi\colon \mathcal{U}(X) \to \mathcal{S}(G)$ by

$$\varphi\colon U \mapsto \langle \max_X(U), U \setminus \max(X) \rangle.$$

The image $\varphi(U)$ is clearly a substructure of G. To see that it is point-preserving, let $l \in U \setminus \max(X)$ and let p be incident to l. By definition, this means that $p > l$, and then since U is an upset, we have $p \in \max_X(U)$. So $\varphi(U)$ is point-preserving.

Now we show that φ is an order-isomorphism. Let U and V be upsets in X. It is easy to see that $U \subseteq V$ implies $\varphi(U) \subseteq \varphi(V)$. Conversely, assume that $\varphi(U) \subseteq \varphi(V)$. Then $\max_X(U) \subseteq \max_X(V)$ and $U \setminus \max(X) \subseteq V \setminus \max(X)$. Since $U = \max_X(U) \cup U \setminus \max(X)$ and similarly for V, it follows that $U \subseteq V$. Hence φ is an order-embedding. Now, for surjectivity, let $H \in \mathcal{S}(G)$. Then there are sets $P_H \subseteq \max(X)$ and $L_H \subseteq X \setminus \max(X)$ such that $H = \langle P_H, L_H \rangle$. Since H is point-preserving, we must have ${\uparrow}L_H \subseteq P_H \cup L_H$, and since ${\uparrow}P_H = P_H$, we have that $P_H \cup L_H$ is an upset in X. Then $\varphi(P_H \cup L_H) = H$, so φ is surjective, and hence φ is an order-isomorphism. Therefore, φ is a double p-algebra isomorphism, and the result holds. □

Corollary 4.3.3. *Let* $\mathbf{A}$ *be a finite regular double p-algebra. Then there exists an incidence structure* G *such that* $\mathbf{A} \cong \mathcal{S}(G)$*.*

Proof. By choosing $X = \mathcal{F}_p(\mathbf{A})$, the finiteness of $\mathbf{A}$ implies that $\mathbf{A} \cong \mathcal{U}(X)$, which is isomorphic to $\mathcal{S}(\mathcal{E}(X))$ by the previous result. □

4.4 Characterising substructure lattices

Before we prove the representation theorem, we need some smaller results that decompose regular double p-algebras into more manageable components. Let $\mathbf{A}$ be a distributive double p-algebra. Recall from Section 4.2 that, for all $a \in \mathbf{A}$, the map $h_a\colon \mathbf{A} \to {\downarrow}a$ is defined by $h_a\colon x \mapsto x \wedge a$. From Lemma 4.2.4, if $a \in \mathrm{Cen}(\mathbf{A})$, then $\langle {\downarrow}a; \vee, \wedge, \neg_a, {\sim_a}, 0, a \rangle$ is a distributive double p-algebra with $\neg_a = \neg x \wedge a$ and ${\sim_a} = {\sim}x \wedge a$. We will identify the set ${\downarrow}a$ with the algebra just described.

Theorem 4.4.1. *Let* $\mathbf{A}$ *be a doubly atomic regular double p-algebra and assume* $\mathbf{A}$ *satisfies* (JID) *and* (MID)*. Define the two sets* B *and* C *by*

$$B = \{b \in \mathcal{A}(\mathbf{A}) \mid (\exists c \in \mathcal{C}(\mathbf{A}))\ b \nleq c\},$$
$$C = \{c \in \mathcal{C}(\mathbf{A}) \mid (\exists a \in \mathcal{A}(\mathbf{A}))\ a \nleq c\}.$$

(1) $\bigvee B$ *and* $\bigwedge C$ *are mutual complements in* $\mathbf{A}$*.*

(2) *$\downarrow\bigvee B \cong \mathcal{P}(B)$.*

(3) *$\downarrow\bigwedge C$ is a doubly atomic complete regular double p-algebra satisfying (JID). Furthermore, for all $a \in \mathcal{A}(\downarrow\bigwedge C)$ and all $c \in \mathcal{C}(\downarrow\bigwedge C)$, we have $a \leq c$.*

Proof. Observe that B is empty if and only if C is empty. If B and C are both empty, then $\bigvee B = 0$ and $\bigwedge C = 1$, so in that case the result is trivial. Assume that B and C are non-empty.

Part (1): Using (JID), we have

$$\bigvee B \wedge \bigwedge C = \bigvee \{b \wedge \bigwedge C \mid b \in B\}.$$

Because elements of B are atoms, either $b \leq \bigwedge C$ or $b \wedge \bigwedge C = 0$, for each $b \in B$. By definition, there exists $c \in C$ with $b \nleq c$, so $b \nleq \bigwedge C$. So for all $b \in B$, we have $b \wedge \bigwedge C = 0$, and therefore $\bigvee B \wedge \bigwedge C = 0$. By a dual argument using (MID), we have $\bigvee B \vee \bigwedge C = 1$, and so $\bigvee B$ and $\bigwedge C$ are mutual complements.

Part (2): To ease our notation, we will write $\neg$ for $\neg_{\bigvee B}$ and $\sim$ for $\sim_{\bigvee B}$. Note that if $x \in \downarrow\bigvee B$, then $\mathcal{A}_{\mathbf{A}}(x) = \downarrow x \cap \mathcal{A}(\mathbf{A}) = \downarrow x \cap \mathcal{A}(\downarrow\bigvee B) = \mathcal{A}_{\downarrow\bigvee B}(x)$, so the notation $\mathcal{A}(x)$ is unambiguous. We will first show that if $x \in \downarrow\bigvee B$, then $\neg x = \bigvee B \backslash \mathcal{A}(x)$. Let $x \in \downarrow\bigvee B$. Firstly, because $\mathbf{A}$ is atomic, $\downarrow\bigvee B$ is also atomic. Hence, by Lemma 4.2.2, we have $\neg x = \neg\bigvee\mathcal{A}(x)$. It now suffices to show that $\neg\bigvee\mathcal{A}(x) = \bigvee B\backslash\mathcal{A}(x)$. From (JID), we get

$$\bigvee \mathcal{A}(x) \wedge \bigvee B\backslash\mathcal{A}(x) = \bigvee\{a \wedge b \mid a \in \mathcal{A}(x) \text{ and } b \in B\backslash\mathcal{A}(x)\} = 0.$$

So $\bigvee B\backslash\mathcal{A}(x) \leq \neg\bigvee\mathcal{A}(x)$. By Lemma 4.2.1, we have $\mathcal{A}(\bigvee B) = B$. Since $x \leq \bigvee B$, we then have $\mathcal{A}(x) \subseteq B$. It follows that $\bigvee\mathcal{A}(x) \vee \bigvee B\backslash\mathcal{A}(x) = \bigvee B$. Because $\bigvee B$ is the top element of $\downarrow\bigvee B$, we have $\bigvee B\backslash\mathcal{A}(x) \geq {\sim}\bigvee\mathcal{A}(x)$, and then since ${\sim}\bigvee\mathcal{A}(x) \geq \neg\bigvee\mathcal{A}(x)$,

$$\bigvee B\backslash\mathcal{A}(x) \leq \neg\bigvee\mathcal{A}(x) \leq {\sim}\bigvee\mathcal{A}(x) \leq \bigvee B\backslash\mathcal{A}(x),$$

which proves that $\neg\bigvee\mathcal{A}(x) = \bigvee B\backslash\mathcal{A}(x)$. We also have $x \vee \bigvee B\backslash\mathcal{A}(x) = \bigvee B$. So

$$\bigvee B\backslash\mathcal{A}(x) \geq {\sim}x \geq \neg x = \bigvee B\backslash\mathcal{A}(x),$$

and hence ${\sim}x = \neg x$. Therefore, $\downarrow\bigvee B$ is a Boolean lattice. It is also complete by the completeness of $\mathbf{A}$, so $\downarrow\bigvee B$ is a complete atomic Boolean lattice with B as its set of atoms, implying $\downarrow\bigvee B \cong \mathcal{P}(B)$.

Part (3): The completeness of $\downarrow\bigwedge C$ follows from the completeness of $\mathbf{A}$, and similarly for (JID). Since regularity is equational by Theorem 1.6.2, it follows that

$\downarrow\bigwedge C$ is regular. Since $\mathbf{A}$ is atomic, we have that $\downarrow\bigwedge C$ is atomic. We now show that $\downarrow\bigwedge C$ is coatomic. Let $x \leq \bigwedge C$. Then $C \subseteq \mathcal{C}_{\mathbf{A}}(x)$. Suppose that $\mathcal{C}_{\mathbf{A}}(x) = C$. By Lemma 4.2.2, we have $\sim x = \sim\bigwedge C$ in $\mathbf{A}$, which is equal to $\bigvee B$ by part (1). It is then easy to check that distributivity is contradicted if $x \neq \bigwedge C$. So we may safely assume that $C \subsetneq \mathcal{C}_{\mathbf{A}}(x)$, and hence there exists $c \in \mathcal{C}_{\mathbf{A}}(x)\backslash C$. We have by (MID) that $c \vee \bigwedge C = \bigwedge\{c \vee d \mid d \in C\} = 1$, so $c \ngeq \bigwedge C$, and then $c \wedge \bigwedge C < \bigwedge C$. A routine calculation shows that if $\bigwedge C$ does not cover $c \wedge \bigwedge C$, then distributivity is contradicted. So $c \wedge \bigwedge C$ is a coatom in $\downarrow\bigwedge C$, and since $x \leq \bigwedge C$, we then have $x \leq c \wedge \bigwedge C$. Observe that every coatom is of this form. Indeed, if x is a coatom in $\downarrow\bigwedge C$, then there exists $c \in \mathcal{C}_{\mathbf{A}}(x)\backslash C$ such that $x \leq c \wedge \bigwedge C < \bigwedge C$, which implies $x = c \wedge \bigwedge C$.

To prove the remainder of (3), let $a \in \mathcal{A}(\downarrow\bigwedge C)$. If $a \in B$, then $a \leq \bigvee B$. Then since $a \leq \bigwedge C$, we have $a \leq \bigvee B \wedge \bigwedge C$. But $\bigvee B \wedge \bigwedge C = 0$ by part (1), so $a \notin B$. Recall that every coatom in $\downarrow\bigwedge C$ is of the form $c \wedge \bigwedge C$, for some $c \in \mathcal{C}(\mathbf{A})\backslash C$. Now let $c \in \mathcal{C}(\mathbf{A})\backslash C$. Because $a \notin B$ implies $a \leq c$, and $a \leq \bigwedge C$ by assumption, it follows that $a \leq c \wedge \bigwedge C$, as required. □

The double p-algebra $\downarrow\bigwedge C$ just seen is the algebra we are going to represent in the next theorem.

Definition 4.4.2. Let $\mathbf{A}$ be a doubly atomic regular double p-algebra. Assume that $\mathbf{A}$ satisfies (JID) and that $a \leq c$, for all $a \in \mathcal{A}(\mathbf{A})$. Let $\mathcal{G}(\mathbf{A})$ be the incidence structure $\langle P, L, I\rangle$ defined by

$$\begin{aligned} P &= \mathcal{A}(\mathbf{A}), \\ L &= \mathcal{C}(\mathbf{A}), \\ I &= \{(a, c) \in P \times L \mid a \leq \sim c\}. \end{aligned}$$

Theorem 4.4.3. *Let $\mathbf{A}$ be a doubly atomic regular double p-algebra. Assume that $\mathbf{A}$ satisfies (JID) and that $a \leq c$, for all $a \in \mathcal{A}(\mathbf{A})$. Then $\mathbf{A}$ is isomorphic to $\mathcal{S}(\mathcal{G}(\mathbf{A}))$. Furthermore, $\mathcal{G}(\mathbf{A})$ has no isolated points and no empty lines.*

Proof. Let $G = \mathcal{G}(\mathbf{A})$ and let $H = \langle P_H, L_H\rangle \in \mathcal{S}(G)$. Define the map $\varphi\colon \mathcal{S}(G) \to A$ by

$$\varphi(H) = \bigvee P_H \vee \bigvee\{\sim c \mid c \in L_H\}.$$

To streamline the proof, we will first show that both $\mathcal{A}(\varphi(H)) = P_H$ and $\mathcal{C}(\varphi(H)) = \mathcal{C}(\mathbf{A})\backslash L_H$. Utilising Lemma 4.2.1, we have

$$\mathcal{A}(\varphi(H)) = \mathcal{A}(\bigvee P_H \vee \bigvee\{\sim c \mid c \in L_H\}) = \mathcal{A}(\bigvee P_H) \cup \mathcal{A}(\bigvee\{\sim c \mid c \in L_H\}).$$

Since H is point preserving, if $c \in L_H$ and $(a,c) \in I$, then $a \in P_H$. In other words, for all $c \in L_H$, we have $\mathcal{A}({\sim}c) \subseteq P_H$. Then by Lemma 4.2.1 and since (JID) holds, it follows that

$$\mathcal{A}(\bigvee\{{\sim}c \mid c \in L_H\}) = \bigcup\{\mathcal{A}({\sim}c) \mid c \in L_H\} \subseteq P_H \subseteq \mathcal{A}(\bigvee P_H).$$

Hence,

$$\mathcal{A}(\varphi(H)) = \mathcal{A}(\bigvee P_H) = \bigcup\{\mathcal{A}(a) \mid a \in P_H\} = P_H.$$

On the other hand, we have

$$\begin{aligned}\mathcal{C}(\varphi(H)) &= \mathcal{C}(\bigvee P_H \vee \bigvee\{{\sim}c \mid c \in L_H\}) \\ &= \mathcal{C}(\bigvee P_H) \cap \mathcal{C}(\bigvee\{{\sim}c \mid c \in L_H\}).\end{aligned}$$

By assumption, if $a \in \mathcal{A}(\mathbf{A})$, then $\mathcal{C}(a) = \mathcal{C}(\mathbf{A})$. So by Lemma 4.2.1, we have $\mathcal{C}(\bigvee P_H) = \bigcap\{\mathcal{C}(p) \mid p \in P_H\} = \mathcal{C}(\mathbf{A})$. Hence, by Lemma 4.2.1 again,

$$\mathcal{C}(\varphi(H)) = \mathcal{C}(\bigvee\{{\sim}c \mid c \in L_H\}) = \bigcap\{\mathcal{C}({\sim}c) \mid c \in L_H\}.$$

Let c and d both be coatoms in $\mathbf{A}$. If $c \neq d$, then $c \vee d = 1$, implying $d \geq {\sim}c$. So $\mathcal{C}({\sim}c) = \mathcal{C}(\mathbf{A})\backslash\{c\}$, for all $c \in L_H$. Therefore,

$$\mathcal{C}(\varphi(H)) = \bigcap\{\mathcal{C}({\sim}c) \mid c \in L_H\} = \bigcap\{\mathcal{C}(\mathbf{A})\backslash\{c\} \mid c \in L_H\} = \mathcal{C}(\mathbf{A})\backslash L_H.$$

Hence, just as claimed, $\mathcal{A}(\varphi(H)) = P_H$ and $\mathcal{C}(\varphi(H)) = \mathcal{C}(\mathbf{A})\backslash L_H$.

Now we will show that φ is an order-isomorphism. For surjectivity, let $x \in A$ and let $H_x = \langle \mathcal{A}(x), \mathcal{C}(\mathbf{A})\backslash\mathcal{C}(x)\rangle$. Clearly H_x is a substructure of G. To show that it is a point-preserving substructure, let $c \in \mathcal{C}(\mathbf{A})\backslash\mathcal{C}(x)$. Then $c \ngeq x$, and so $c \vee x = 1$, implying $x \geq {\sim}c$. If $(a,c) \in I$, then $a \leq {\sim}c \leq x$, and it follows that $a \in \mathcal{A}(x)$. Hence H_x is point-preserving. We then have $\mathcal{A}(\varphi(H_x)) = \mathcal{A}(x)$ and $\mathcal{C}(\varphi(H_x)) = \mathcal{C}(\mathbf{A})\backslash(\mathcal{C}(\mathbf{A})\backslash\mathcal{C}(x)) = \mathcal{C}(x)$. So $\varphi(H_x) = x$ by Lemma 4.2.2, proving surjectivity. Next, we show that φ is an order-embedding. Let $H_1, H_2 \in \mathcal{S}(G)$, and write $H_i = \langle P_i, L_i\rangle$, for each $i \in \{1,2\}$. If $H_1 \leq H_2$, then $P_1 \subseteq P_2$ and $\{{\sim}c \mid c \in L_1\} \subseteq \{{\sim}c \mid c \in L_2\}$, implying $\varphi(H_1) \leq \varphi(H_2)$. Conversely, assume that $\varphi(H_1) \leq \varphi(H_2)$. Then $\mathcal{A}(\varphi(H_1)) \subseteq \mathcal{A}(\varphi(H_2))$, so $P_1 \subseteq P_2$. Similarly, we have $\mathcal{C}(\varphi(H_2)) \subseteq \mathcal{C}(\varphi(H_1))$, so $\mathcal{C}(\mathbf{A})\backslash L_2 \subseteq \mathcal{C}(\mathbf{A})\backslash L_1$, and it follows that $L_1 \subseteq L_2$. Thus $H_1 \leq H_2$, and hence φ is an order-embedding. We conclude that φ is an order-isomorphism, and therefore φ is a double p-algebra isomorphism.

To show that G has no isolated points, suppose otherwise. Then there is a point $p \in P$ such that $c = \langle P\backslash\{p\}, L\rangle$ is a coatom in $\mathcal{S}(G)$. But $a = \langle\{p\}, \varnothing\rangle$ is an atom with $a \nleq c$, contradicting the fact that $\mathcal{S}(G)$ is isomorphic to $\mathbf{A}$. The reasoning is similar to show that G contains no empty lines. □

Corollary 4.4.4. *Let* $\mathbf{A}$ *be a doubly atomic regular double p-algebra and assume that* $\mathbf{A}$ *satisfies* (JID) *and* (MID). *Then there is a set* B *and an incidence structure* G *with no isolated points and no empty lines such that* $\mathbf{A} \cong \mathcal{P}(B) \times \mathcal{S}(G)$.

Proof. Let B and C be defined as in Theorem 4.4.1. By Lemma 4.2.5 we have that $\mathbf{A} \cong {\downarrow}\bigvee B \times {\downarrow}\bigwedge C \cong \mathcal{P}(B) \times {\downarrow}\bigwedge C$. By Theorem 4.4.3, there is an incidence structure G with no isolated points and no empty lines such that ${\downarrow}\bigwedge C \cong \mathcal{S}(G)$, so $\mathbf{A} \cong \mathcal{P}(B) \times \mathcal{S}(G)$, as claimed. □

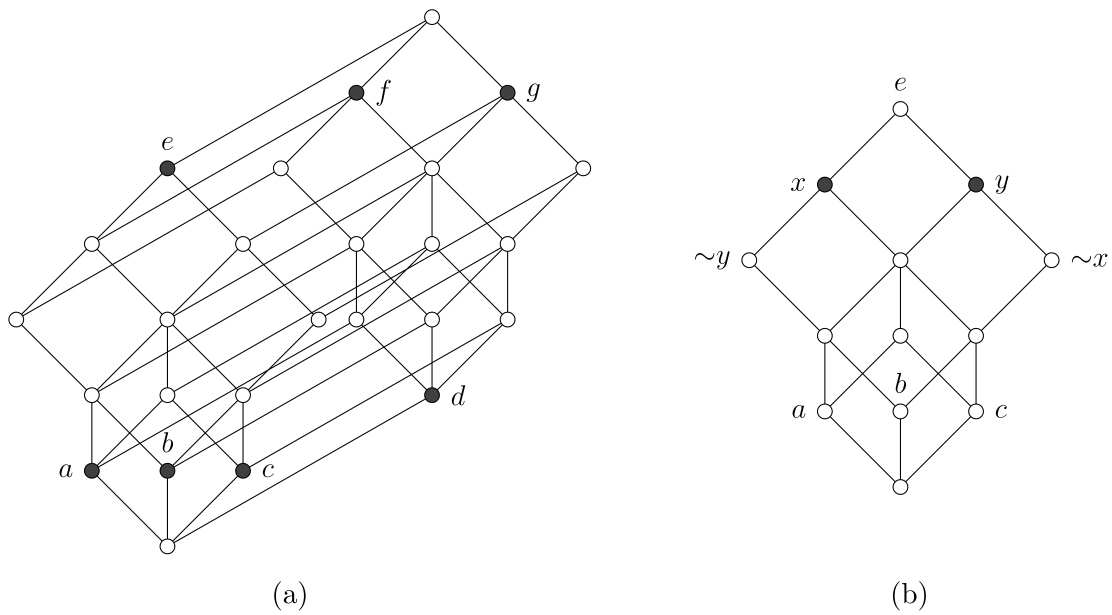


Figure 4.3

We now give the characterisation in full.

Theorem 4.4.5. *Let* $\mathbf{A}$ *be a regular double p-algebra. The following are equivalent:*

(1) $\mathbf{A} \cong \mathcal{P}(B) \times \mathcal{S}(G)$, *for some set* B *and some incidence structure* G *with no isolated points and no empty lines;*

(2) $\mathbf{A} \cong \mathcal{P}(B) \times \mathcal{S}(G)$, *for some set* B *and some incidence structure* G;

(3) $\mathbf{A} \cong \mathcal{S}(G)$, *for some incidence structure* G;

(4) $\mathbf{A}$ *is completely distributive and doubly atomic;*

(5) $\mathbf{A}$ *satisfies* (JID) *and* (MID) *and is doubly atomic.*

Proof. (1) ⇒ (2) is immediate, and (2) ⇒ (3) follows from Lemma 4.1.6. Theorem 4.1.5 proves (3) ⇒ (4), and (4) ⇒ (5) is obvious. We have just seen the proof of (5) ⇒ (1), and this closes the cycle. □

Observe that Corollary 4.3.3 also follows from this result. To finish this chapter, we will illustrate the construction with an example. Let $\mathbf{A}$ be the regular double p-algebra given in Figure 4.3a. The atoms of $\mathbf{A}$ are the elements $\{a, b, c, d\}$, and the coatoms are the elements $\{e, f, g\}$, as labelled. Recall the notation of Theorem 4.4.1:

$$B = \{b \in \mathcal{A}(\mathbf{A}) \mid (\exists c \in \mathcal{C}(\mathbf{A}))\ b \nleq c\},$$
$$C = \{c \in \mathcal{C}(\mathbf{A}) \mid (\exists a \in \mathcal{A}(\mathbf{A}))\ a \nleq c\}.$$

We have $B = \{d\}$, because every atom except d is below every coatom but $d \nleq e$. Similarly, we have $C = \{e\}$. So $\bigvee B = d$ and $\bigwedge C = e$. Clearly $\downarrow d$ is just a two-element chain, so let us represent that by the two-element Boolean algebra $\mathbf{2}$. Thus, by Theorem 4.4.1, we have $\mathbf{A} \cong \mathbf{2} \times \downarrow e$. We have drawn the lattice $\downarrow e$ in Figure 4.3b. Its atoms are $\{a, b, c\}$, and its coatoms are $\{x, y\}$, as labelled. Now we apply Theorem 4.4.3 to $\downarrow e$. The set of points is $P = \{a, b, c\}$ and the set of lines is $L = \{x, y\}$. Recall that the incidence relation is given by

$$I = \{(a, c) \in P \times L \mid a \leq {\sim}c\}.$$

Now observe that $b, c \in \downarrow{\sim}x$ but $a \nleq {\sim}x$, and $a, b \in \downarrow{\sim}y$ but $c \nleq {\sim}y$. So the two lines are incident to two vertices each, resulting in the graph G drawn in Figure 4.4a. We then have $\downarrow e \cong \mathcal{S}(G)$. Since $\mathbf{A} \cong \mathbf{2} \times \downarrow e$, we conclude that $\mathbf{A}$ is isomorphic to $\mathcal{S}(G')$, where G' is the incidence structure obtained from G by adding the isolated point d (see Figure 4.4b).

(a) The graph G (b) The graph G'

Figure 4.4

5

Expansions of Heyting algebras

Since congruences on a Heyting algebra are determined by filters, if $\mathbf{A}$ is an algebra with a Heyting algebra reduct, then every congruence on $\mathbf{A}$ is of the form $\theta(F)$, for some filter F. It is then of natural interest to characterise the filters that correspond to congruences on $\mathbf{A}$. As discussed in Chapter 1, Köhler characterised these filters for double Heyting algebras, and Sankappanavar extended it to dually pseudocomplemented Heyting algebras. Specifically, congruences on either of those algebras are determined exactly by filters closed under the operation $x \mapsto \neg{\sim}x$.

This bears some resemblance to the case for Boolean algebras with operators. An algebra $\mathbf{B} = \langle B; \vee, \wedge, \neg, \{f_i \mid i \in I\}, 0, 1\rangle$ is a *Boolean algebra with* (*dual*) *operators* (BAO for short) if $\langle B; \vee, \wedge, \neg, 0, 1\rangle$ is a Boolean algebra and, for each $i \in I$, the operation f_i is a unary map satisfying $f_i 1 = 1$ and $f_i(x \wedge y) = f_i x \wedge f_i y$. If I is finite, then congruences on $\mathbf{B}$ are determined by filters closed under the map d, defined by

$$dx = \bigwedge \{f_i x \mid i \in I\}.$$

This is easily generalised to operators of finite arity (see Jipsen [51] for example). The overall aim of this chapter is to thread together the theory of BAOs and the theory of double Heyting algebras. In general, a unary term defining congruences on an expanded Heyting algebra will be called a *congruence-filter term*. We will use a construction due to Hasimoto [42] to construct congruence-filter terms under certain conditions, and then prove Sankappanavar and Köhler's results as corollaries.

Except for Section 5.5 and those that are otherwise attributed, the results of this chapter are adapted from the author's paper published in *Studia Logica* [86] and are entirely the original work of the author. Note that in [86], our wording is different compared to this thesis: in that paper, the terminology was "normal filter" and "normal filter term", whereas here, we use the expressions "congruence-filter", "congruence-filter term", and "compatibility term". The proofs in Section 5.5 are due to Tomasz Kowalski. They are not currently published, but they are currently included in a manuscript under preparation by Davey, Kowalski, and the author.

5.1 Congruence-filters and compatibility terms

In this chapter, the letters i, j, k, m and n will always denote non-negative integers. We will not consider nullary operations here, so when we say "let $f\colon A^n \to A$ be a map" or "let f be an n-ary operation on A", we will assume implicitly that $n \geq 1$. There is no technical reason for this, it simply eases notation: because constants do not change congruences, the results still apply with nullary operations in the signature.

Definition 5.1.1. We will say that an algebra $\mathbf{A} = \langle A; M, \vee, \wedge, \to, 0, 1\rangle$ is an *expanded Heyting algebra* (EHA for short) if $\langle A; \vee, \wedge, \to, 0, 1\rangle$ is a Heyting algebra and M is a set of operations on A. Throughout this chapter, the set M will be a fixed but arbitrary set of operations. We will typically use the symbol K for a set of operations that are not necessarily in the signature of an algebra.

Let $f\colon A^n \to A$ be a map. We say that a filter $F \subseteq A$ is *compatible with* f if the following implication is satisfied, for all $x_1, y_1, \dots, x_n, y_n \in A$:

$$x_1 \leftrightarrow y_1, \dots, x_n \leftrightarrow y_n \in F \implies f(x_1, \dots, x_n) \leftrightarrow f(y_1, \dots, y_n) \in F.$$

If K is a set of operations on A, then we say that a filter is *compatible with* K if it is compatible with every operation in K. The correspondence between filters and Heyting algebra congruences ensures that every filter is compatible with the Heyting algebra operations. Recall that if F is a filter of $\mathbf{A}$, then $\theta(F)$ is the Heyting algebra congruence defined by

$$\theta(F) = \{(x, y) \in A^2 \mid x \leftrightarrow y \in F\}.$$

We say that a filter F is a *congruence-filter* (*of* $\mathbf{A}$) if $\theta(F)$ is a congruence on $\mathbf{A}$. Let $\mathrm{Fil}(\mathbf{A})$ denote the set of congruence-filters of $\mathbf{A}$. It is easily verified that $\mathrm{Fil}(\mathbf{A})$, ordered by set inclusion, is a complete lattice, so we will let $\mathbf{Fil}(\mathbf{A})$ denote the lattice of congruence-filters of $\mathbf{A}$. For all $x \in A$, the congruence-filter generated by x is denoted by $\mathrm{Fg}^{\mathbf{A}}(x)$.

The next result follows by definition, but is properly attributed to Hasimoto. We take this moment to note that in [42], Hasimoto unconventionally uses the word "operator" for an arbitrary operation on a Heyting algebra.

Theorem 5.1.2 (Hasimoto [42])**.** *Let* $\mathbf{A}$ *be an* EHA*. A filter of* $\mathbf{A}$ *is a congruence-filter if and only if it is compatible with* M*. The map* $\theta\colon \mathbf{Fil}(\mathbf{A}) \to \mathbf{Con}(\mathbf{A})$ *given by* $F \mapsto \theta(F)$ *is a lattice isomorphism with its inverse given by* $\alpha \mapsto 1/\alpha$*.*

Definition 5.1.3. Let $\mathbf{A}$ be an EHA and let K be a set of operations on A, not necessarily in its signature. A unary term t in the language of $\mathbf{A}$ will be called a *compatibility term for* K if

(1) $t^{\mathbf{A}}$ is order-preserving,

(2) for every $F \subseteq A$, if F is a filter of $\mathbf{A}$, then F is compatible with K if and only if F is closed under $t^{\mathbf{A}}$.

It is easily seen that the filter $\{1\}$ is compatible with any arbitrarily chosen operation. It follows that if t is a compatibility term, then we must always have $t^{\mathbf{A}}1 = 1$. If $K = \{f\}$, then we will omit the braces and say that t is a *compatibility term for* f. If t is a compatibility term for the set M, then we say that t is a *congruence-filter term on* $\mathbf{A}$. Equivalently, t is a congruence-filter term on $\mathbf{A}$ if condition (2) above is replaced with the following:

(2′) for every $F \subseteq A$, if F is a filter of $\mathbf{A}$, then F is a congruence-filter of $\mathbf{A}$ if and only if F is closed under $t^{\mathbf{A}}$.

An algebra $\mathbf{A}$ *has a congruence-filter term* if there exists a congruence-filter term on $\mathbf{A}$. If $\mathcal{K}$ is a class of EHAs with a common signature and t is a term in the language of $\mathcal{K}$, we say that t is a *congruence-filter term on* $\mathcal{K}$ provided that t is a congruence-filter term on every algebra in $\mathcal{K}$. We say that $\mathcal{K}$ *has a congruence-filter term* if there exists a congruence-filter term on $\mathcal{K}$.

Remark 5.1.4. In order to ease notation, for the rest of this thesis, we will not distinguish between terms and term functions, unless the distinction is necessary.

The proof of the following lemma is straightforward, and it is the reason we will direct much of our focus towards compatibility terms for a single operation at a time.

Lemma 5.1.5. *Let* $\mathbf{A}$ *be an* EHA *and let* K_1 *and* K_2 *be sets of operations on* A*. If* t_1 *is a compatibility term for* K_1 *and* t_2 *is a compatibility term for* K_2*, then the term* t *defined by* $tx = t_1x \wedge t_2x$ *is a compatibility term for* $K_1 \cup K_2$*.*

In particular, if M is finite, then to find a congruence-filter term it would be sufficient to find a compatibility term for each operation in M. Our primary example of a congruence-filter term is the term $\neg{\sim}x$ for double Heyting algebras from Corollary 1.5.7. Our next step in the investigation is to find more general methods for constructing compatibility terms.

5.2 Constructing compatibility terms

Starting with work by Hasimoto, we will see that, if it is in the signature, a unary normal operator provides its own compatibility term. Lemma 5.1.5 then implies that an EHA with finitely many unary operators in its signature has a congruence-filter term, generalising the standard result for BAOs. We will begin by considering operators of arbitrary finite arity.

Warning. We will be generalising some standard concepts on BAOs. The high degree of symmetry of Boolean algebras means that results on join-preserving operations can be readily dualised to results on meet-preserving operations. This is not the case for Heyting algebras. Conventionally, an operator on a Boolean algebra is additive, i.e., join-preserving (see Goldblatt [38] or Jónsson [53] for example), but meet-preserving operations are more natural in the current setting. Thus, the reader is warned that what we call an operator is actually a dual operator in the traditional sense.

Definition 5.2.1. Let $\mathbf{A}$ be a Heyting algebra and let f be an n-ary operation on A. For all $a \in A$ and all $k \leq n$, let $f_k(a)$ be the $(n-1)$-ary operation given by

$$f_k(a)(x_1, \dots, x_{n-1}) = f(x_1, \dots, x_{k-1}, a, x_k, \dots, x_{n-1}).$$

The map f is an *operator* if, for all $k \leq n$ and all $x_1, \dots, x_{n-1}, y, z \in A$,

$$f_k(y \wedge z)(x_1, \dots, x_{n-1}) = f_k(y)(x_1, \dots, x_{n-1}) \wedge f_k(z)(x_1, \dots, x_{n-1}),$$

and f is *normal* if, for all $k \leq n$ and all $x_1, \dots, x_{n-1} \in A$,

$$f_k(1)(x_1, \dots, x_{n-1}) = 1.$$

The map f is *order-preserving* if it preserves the pointwise order on $\mathbf{A}^n$. Note that a unary operation $g\colon A \to A$ is a normal operator if and only if g preserves $\wedge$ and satisfies $g1 = 1$. It is easy to see that an operator is order-preserving.

Lemma 5.2.2 (Hasimoto [42]). *Let $\mathbf{A}$ be a Heyting algebra, let f be a unary normal operator on A, and let F be a filter of $\mathbf{A}$. Then F is compatible with f if and only if F is closed under f.*

Thus, if a unary operator f is in the signature of an EHA, then f is a compatibility term for f. Next, we will introduce a construction of Hasimoto, which will be useful for characterising congruence-filters in some cases. Let $\mathbf{A}$ be an EHA and let f be an n-ary operation on A. For each $a \in A$, define the set $f^{\leftrightarrow}(a)$ by

$$f^{\leftrightarrow}(a) = \{f(b_1, \dots, b_n) \leftrightarrow f(c_1, \dots, c_n) \mid (\forall j \leq n)\ b_j, c_j \in A \text{ and } a \leq b_j \leftrightarrow c_j\}.$$

For any set K of operations on A, define the unary partial operation $[K]$ on A by

$$[K]a = \bigwedge \bigcup \{f^{\leftrightarrow}(a) \mid f \in K\},$$

if the infimum exists, and undefined otherwise. If $K = \{f\}$, we will write $[f]$ instead. We say that $[K]$ *exists in* $\mathbf{A}$ if $[K]a$ is defined, for all $a \in A$.

Lemma 5.2.3 (Hasimoto [42])**.** *Let* $\mathbf{A}$ *be a Heyting algebra and let* K *be a set of operations on* A*. If* $[K]$ *exists in* $\mathbf{A}$*, then* $[K]$ *is a normal operator and* $[[K]] = [K]$*.*

Because $[K]$ is a normal operator when it exists, we will sometimes describe this construction using the word *normalisation*. To give an idea how the construction works, let $\mathbf{A}$ be an EHA, let $a \in A$, and recall that M is the set of additional fundamental operations on $\mathbf{A}$. If f is in the signature of $\mathbf{A}$, then the set $f^{\leftrightarrow}(a)$ contains elements that must be in any congruence-filter containing a. If the infimum of $f^{\leftrightarrow}(a)$ exists, then that element encapsulates some portion of the congruence-filter generated by a. If $[M]a$ is defined, then it is the infimum of all such elements, so its upset contains the congruence-filter generated by a. Hasimoto has characterised when $[M]$ determines every congruence on $\mathbf{A}$. The characterisation is given below in Theorem 5.2.5.

Definition 5.2.4. Let $\mathbf{A}$ be an EHA and assume $[M]$ exists in $\mathbf{A}$. We let $[\mathbf{A}]$ denote the EHA with underlying set A and a single operation $[M]$, i.e.,

$$[\mathbf{A}] = \langle A; \vee, \wedge, \rightarrow, [M], 0, 1\rangle.$$

Theorem 5.2.5 (Hasimoto [42])**.** *Let* $\mathbf{A}$ *be an EHA and assume* $[M]$ *exists in* $\mathbf{A}$*.*

(1) $\mathrm{Fil}([\mathbf{A}]) \subseteq \mathrm{Fil}(\mathbf{A})$.

(2) $\mathrm{Con}([\mathbf{A}]) \subseteq \mathrm{Con}(\mathbf{A})$.

(3) *The following are equivalent:*

(i) $[M]a \in \mathrm{Fg}^{\mathbf{A}}(a)$*, for all* $a \in A$;

(ii) $\mathbf{Fil}(\mathbf{A}) = \mathbf{Fil}([\mathbf{A}])$;

(iii) $\mathbf{Con}(\mathbf{A}) = \mathbf{Con}([\mathbf{A}])$.

Importantly, Hasimoto's construction produces a partial operation, and the characterisation of congruences need not always apply. Even if it does apply, there is no guarantee that $[M]$ will be a term function on the algebra. But if $\mathbf{A}$ is an EHA and $[M]$ exists in $\mathbf{A}$, it follows from Lemma 5.2.2 and Lemma 5.2.3 that $[M]$ is a congruence-filter term on the normalised algebra $[\mathbf{A}]$. This is a special case of the following lemma.

Lemma 5.2.6. *Let* $\mathbf{A}$ *be an* EHA *and assume* $[M]$ *exists in* $\mathbf{A}$*. If there is a term* t *in the language of* $\mathbf{A}$ *such that* $tx = [M]x$*, then* t *is a congruence-filter term on* $\mathbf{A}$*.*

Proof. Let t be such a term. Since $[M]$ is a normal operator by Lemma 5.2.3, it follows that t is order-preserving. It remains to show that a filter $F \subseteq A$ is a congruence-filter of $\mathbf{A}$ if and only if F is closed under t. If F is closed under t, it follows from Lemma 5.2.2 that F is compatible with $[M]$. So F is a congruence-filter of $\mathbf{A}$ by Theorem 5.2.5. Conversely, assume F is a congruence-filter of $\mathbf{A}$ and let $x \in F$. By definition, we then have $(x, 1) \in \theta(F)$, and so $(tx, t1) \in \theta(F)$. Since $[M]$ is normal, we have $(tx, 1) \in \theta(F)$, and so $tx \in 1/\theta(F) = F$. □

We will now see some sufficient conditions for $[K]$ to exist and some conditions enabling us to apply Lemma 5.2.6. The first two conditions are due to Hasimoto.

Definition 5.2.7. Let $\mathbf{A}$ be Heyting algebra, let f be an n-ary operation on A, and let $a \in A$. For each $k \leq n$, let $f^{(k)}a$ be an abbreviation for $f_k(a)(0, \ldots, 0)$. That is,

$$f^{(k)}a = f(0, \ldots, 0, a, 0, \ldots, 0),$$

where a is in the k-th position. Define the sets $f^{\rightarrow}(a)$ and $f^{\leftarrow}(a)$ by

$$f^{\rightarrow}(a) = \{f(b_1, \ldots, b_n) \rightarrow f(a \wedge b_1, \ldots, a \wedge b_n) \mid b_1, \ldots, b_n \in A\},$$
$$f^{\leftarrow}(a) = \{f(a \wedge b_1, \ldots, a \wedge b_n) \rightarrow f(b_1, \ldots, b_n) \mid b_1, \ldots, b_n \in A\}.$$

Lemma 5.2.8 (Hasimoto [42])**.** *Let* $\mathbf{A}$ *be a Heyting algebra, let* f *be an* n*-ary operation on* A*, and let* $a, x \in A$*. If* f *is order-preserving, then* x *is a lower bound of* $f^{\leftrightarrow}(a)$ *if and only if* x *is a lower bound of* $f^{\rightarrow}(a)$*. In particular, should either of* $\bigwedge f^{\leftrightarrow}(a)$ *or* $\bigwedge f^{\rightarrow}(a)$ *exist, then the other exists and they are equal, i.e.,*

$$[f]a = \bigwedge f^{\rightarrow}(a).$$

The previous lemma does not guarantee that $[f]$ exists, but the next one does.

Lemma 5.2.9 (Hasimoto [42])**.** *Let* $\mathbf{A}$ *be a Heyting algebra and let* f *be an* n*-ary operation on* A*. If* f *is a normal operator, then* $[f]$ *exists in* $\mathbf{A}$ *and, for all* $a \in A$*,*

$$[f]a = \bigwedge \{f^{(k)}a \mid k \leq n\}.$$

For example, if f is ternary, then $[f]a = f(a, 0, 0) \wedge f(0, a, 0) \wedge f(0, 0, a)$.

Corollary 5.2.10. *If* $\mathbf{A}$ *is an* EHA *of finite signature and each operation in* M *is a normal operator, then* $\mathbf{A}$ *has a congruence-filter term given by*

$$tx = \bigwedge \{[f]x \mid f \in M\}.$$

This is our first general guarantee of possessing a congruence-filter term. In particular, it includes the Boolean algebras with operators mentioned in the introduction to this chapter. Importantly, the proof of the above result is not dualisable to join-preserving operations. Rather, we turn to meet-reversing operations, permitting a slight tweak to the result.

Definition 5.2.11. Let $\mathbf{A}$ be a Heyting algebra and let f be an n-ary operation on A. We say that f is an *anti-operator* if, for all $k \leq n$ and all $x_1, \dots, x_{n-1}, y, z \in A$,

$$f_k(y \wedge z)(x_1, \dots, x_{n-1}) = f_k(y)(x_1, \dots, x_{n-1}) \vee f_k(z)(x_1, \dots, x_{n-1}),$$

and f is *anti-normal* if, for all $k \leq n$ and all $x_1, \dots, x_{n-1} \in A$,

$$f_k(1)(x_1, \dots, x_{n-1}) = 0.$$

For convenience, we will call an anti-normal anti-operator an *anti-normal operator*. The map f is *order-reversing* if it reverses the pointwise order on $\mathbf{A}^n$. It is easy to see that an anti-operator is order-reversing.

The dual pseudocomplement operation is an example of an anti-normal operator.

Lemma 5.2.12. *Let $\mathbf{A}$ be a Heyting algebra, let f be an n-ary operation on A, and let $a, x \in A$. If f is order-reversing, then x is a lower bound of $f^{\leftrightarrow}(a)$ if and only if x is a lower bound of $f^{\leftarrow}(a)$. In particular, should either of $\bigwedge f^{\leftrightarrow}(a)$ or $\bigwedge f^{\leftarrow}(a)$ exist, then the other exists and they are equal, i.e.,*

$$[f]a = \bigwedge f^{\leftarrow}(a).$$

Proof. Assume f is order-reversing and let x be a lower bound of $f^{\leftrightarrow}(a)$. For all $b_1, \dots, b_n \in A$ and all $i \leq n$, we have $a \leq b_i \to (a \wedge b_i) = (a \wedge b_i) \leftrightarrow b_i$, and so

$$f(a \wedge b_1, \dots, a \wedge b_n) \leftrightarrow f(b_1, \dots, b_n) \in f^{\leftrightarrow}(a).$$

Since $b_i \geq a \wedge b_i$, we have $f(a \wedge b_1, \dots, a \wedge b_n) \geq f(b_1, \dots, b_n)$, and therefore

$$f(a \wedge b_1, \dots, a \wedge b_n) \leftrightarrow f(b_1, \dots, b_n) = f(a \wedge b_1, \dots, a \wedge b_n) \to f(b_1, \dots, b_n).$$

So $f^{\leftarrow}(a) \subseteq f^{\leftrightarrow}(a)$, and hence x is a lower bound of $f^{\leftarrow}(a)$. Conversely, let x be a lower bound of $f^{\leftarrow}(a)$, let $b_1, c_1, \dots, b_n, c_n \in A$, and assume $a \leq b_i \leftrightarrow c_i$, for all $i \leq n$. Then, for each $i \leq n$, we have $a \wedge b_i \leq c_i$ and $a \wedge c_i \leq b_i$. Because f is order-reversing, we have $f(a \wedge c_1, \dots, a \wedge c_n) \geq f(b_1, \dots, b_n)$. Then, since $\to$ is order-reversing in the first coordinate, this implies that

$$f(b_1, \dots, b_n) \to f(c_1, \dots, c_n) \geq f(a \wedge c_1, \dots, a \wedge c_n) \to f(c_1, \dots, c_n) \geq x.$$

Similarly, we have $f(c_1, \dots, c_n) \to f(b_1, \dots, b_n) \geq x$, and therefore

$$f(b_1, \dots, b_n) \leftrightarrow f(c_1, \dots, c_n) \geq x,$$

as required. □

Lemma 5.2.13. *Let* $\mathbf{A}$ *be a Heyting algebra and let* f *be an* n*-ary operation on* A*. If* f *is an anti-normal operator, then* $[f]$ *exists in* $\mathbf{A}$ *and, for all* $a \in A$*,*

$$[f]a = \bigwedge\{\neg f^{(k)}a \mid k \leq n\}.$$

Proof. Assume that f is an anti-normal operator and let $a \in A$. We will show that $\bigwedge\{\neg f^{(k)}a \mid k \leq n\}$ is the infimum of $f^{\leftarrow}(a)$, and then, since an anti-operator is order-reversing, the result follows from Lemma 5.2.12. Let $b_1, \dots, b_n \in A$. An elementary induction argument shows that

$$f(a \wedge b_1, \dots, a \wedge b_n) = \bigvee \big\{ f(z_1, \dots, z_n) \mid (\forall i \leq n)\ z_i \in \{a, b_i\} \big\}.$$

Then, since $\to$ reverses joins in the first coordinate, we have

$$\begin{aligned} &f(a \wedge b_1, \dots, a \wedge b_n) \to f(b_1, \dots, b_n) \\ &= \Big[\bigvee \big\{ f(z_1, \dots, z_n) \mid z_i \in \{a, b_i\} \big\} \Big] \to f(b_1, \dots, b_n) \\ &= \bigwedge \big\{ f(z_1, \dots, z_n) \to f(b_1, \dots, b_n) \mid z_i \in \{a, b_i\} \big\}. \end{aligned}$$

If $z_i = b_i$, for all $i \leq n$, then $f(z_1, \dots, z_n) \to f(b_1, \dots, b_n) = 1$. Otherwise, there is at least one $i \leq n$ such that $z_i = a$. Since f is order-reversing, we have, for all $x_1, \dots, x_n \in A$ and all $k \leq n$,

$$f(x_1, \dots, x_{k-1}, a, x_{k+1}, \dots, x_n) \leq f(0, \dots, 0, a, 0, \dots, 0) = f^{(k)}a.$$

Then, since $\to$ is order-reversing in the first coordinate, we have

$$\begin{aligned} &\bigwedge \big\{ f(z_1, \dots, z_n) \to f(b_1, \dots, b_n) \mid z_i \in \{a, b_i\} \big\} \\ &\geq \bigwedge \big\{ f^{(k)}(a) \to f(b_1, \dots, b_n) \mid k \leq n \big\}. \end{aligned}$$

Furthermore, since $\to$ is order-preserving in the second coordinate, we have that $f^{(k)}(a) \to f(b_1, \dots, b_n) \geq \neg f^{(k)}a$, and it follows that $\bigwedge\{\neg f^{(k)}a \mid k \leq n\}$ is a lower bound of $f^{\leftarrow}(a)$. To complete the proof, it suffices to show that $\neg f^{(k)}a \in f^{\leftarrow}(a)$, for all $k \leq n$. It will then follow that $\bigwedge\{\neg f^{(k)}a \mid k \leq n\}$ is the infimum of $f^{\leftarrow}(a)$. Let $k \leq n$, let $b_k = 1$, and for all $i \neq k$, let $b_i = 0$. Then

$$f(a \wedge b_1, \dots, a \wedge b_n) \to f(b_1, \dots, b_n) = f^{(k)}a \to f^{(k)}1,$$

and since $f^{(k)}1 = 0$, this is equal to $\neg f^{(k)}a$. Therefore, $\neg f^{(k)}a \in f^{\leftarrow}(a)$, and hence $[f]a = \bigwedge\{\neg f^{(k)}a \mid k \leq n\}$. □

Definition 5.2.14. Let $\mathbf{A}$ be an EHA. If every operation in M is either a normal operator or an anti-normal operator, we will say that $\mathbf{A}$ is a *Heyting algebra with operators* (HAO for short).

Combining Lemma 5.1.5, Lemma 5.2.6, Lemma 5.2.9, and Lemma 5.2.13 then yields the following result, extending the standard construction for BAOs mentioned in the introduction.

Corollary 5.2.15. *Let $\mathbf{A}$ be an HAO. If $\mathbf{A}$ is of finite signature, then $\mathbf{A}$ has a congruence-filter term. More precisely, if n denotes the arity of f and $f \in M$, we have*

$$[f]x = \begin{cases} f^{(1)}(x) \wedge f^{(2)}(x) \wedge \ldots \wedge f^{(n)}(x), & \text{if } f \text{ is normal,} \\ \neg f^{(1)}(x) \wedge \neg f^{(2)}(x) \wedge \ldots \wedge \neg f^{(n)}(x), & \text{if } f \text{ is anti-normal,} \end{cases}$$

and then

$$tx = \bigwedge_{f \in M} [f]x$$

is a congruence-filter term on $\mathbf{A}$.

Because the dual pseudocomplement is an anti-normal operator, it follows that $[\sim] = \neg\sim$ on every H$^+$-algebra. So by Corollary 5.2.15, congruence-filters of an H$^+$-algebra are exactly the filters closed under $\neg\sim$. Hence we obtain Theorem 1.4.4 as a corollary. From Theorem 1.5.6, we obtain the same result for double Heyting algebras. To polish off this section, we give a direct proof for double Heyting algebras using the normalisation technique. We do not have a more general proof.

Lemma 5.2.16. *Let $\mathbf{A}$ be a double Heyting algebra. Then $[\dot{-}]$ exists in $\mathbf{A}$ and, for all $a \in A$, we have $[\dot{-}]a = \neg\sim a$.*

Proof. Let $a \in A$. We will prove that $\neg\sim a$ is the minimum of $\dot{-}^{\leftrightarrow}(a)$. Firstly, we have

$$(1 \dot{-} a) \leftrightarrow (1 \dot{-} 1) = \sim a \leftrightarrow 0 = \neg\sim a,$$

and since $1 \leftrightarrow 1 = 1$ and $a \leftrightarrow 1 = a$, we have $\neg\sim a \in \dot{-}^{\leftrightarrow}(a)$ by definition. Now let $x_1 \leftrightarrow x_2 \geq a$ and let $y_1 \leftrightarrow y_2 \geq a$. We will prove that

$$\neg\sim a \wedge (x_1 \dot{-} y_1) = \neg\sim a \wedge (x_2 \dot{-} y_2),$$

from which it will then follow by Lemma 1.2.7 that $\neg\sim a \leq (x_1 \dot{-} y_1) \leftrightarrow (x_2 \dot{-} y_2)$. Since $x_1 \leftrightarrow x_2 \geq a$, we have $x_1 \wedge a = x_2 \wedge a$, and it follows by distributivity that

$x_1 \vee {\sim}a = x_2 \vee {\sim}a$. Similarly, $y_1 \vee {\sim}a = y_2 \vee {\sim}a$. So by Theorem 1.5.2, we have

$$\begin{aligned} {\sim}a \vee (x_1 \dot{-} y_1) &= {\sim}a \vee [({\sim}a \vee x_1) \dot{-} ({\sim}a \vee y_1)] \\ &= {\sim}a \vee [({\sim}a \vee x_2) \dot{-} ({\sim}a \vee y_2)] \\ &= {\sim}a \vee (x_2 \dot{-} y_2). \end{aligned}$$

Applying $\neg$ to both sides shows that $\neg{\sim}a \wedge \neg(x_1 \dot{-} y_1) = \neg{\sim}a \wedge \neg(x_2 \dot{-} y_2)$, as required. □

5.3 Subdirectly irreducible and simple algebras

Being able to concisely describe congruences using a single unary term has clear advantages. Before we characterise subdirectly irreducible algebras, we first need an extra condition on congruence-filter terms.

Definition 5.3.1. Let $\mathcal{V}$ be a variety of EHAs and let t be a unary term in the language of $\mathcal{V}$. We will say that t is *descending* if $\mathcal{V} \models tx \leq x$.

If t is a congruence-filter term, then the term d given by $dx = x \wedge tx$ is a descending congruence-filter term. Thus, a variety has a congruence-filter term if and only if it has a descending congruence-filter term. This means we suffer no loss of generality by assuming that every congruence-filter term is descending.

Theorem 5.3.2. *Let* $\mathbf{A}$ *be an EHA, let* $x, y, z, w \in A$*, and assume that* d *is a descending congruence-filter term on* $\mathbf{A}$.

(1) $\mathrm{Fg}^{\mathbf{A}}(x) = {\uparrow}\{d^n x \mid n \in \omega\}$.

(2) $\mathrm{Cg}^{\mathbf{A}}(x, y) = \theta(\mathrm{Fg}^{\mathbf{A}}(x \leftrightarrow y))$.

(3) *The following are equivalent:*

- (i) $(w, z) \in \mathrm{Cg}^{\mathbf{A}}(x, y)$;
- (ii) $(w \leftrightarrow z, 1) \in \mathrm{Cg}^{\mathbf{A}}(x \leftrightarrow y, 1)$;
- (iii) $w \leftrightarrow z \in \mathrm{Fg}^{\mathbf{A}}(x \leftrightarrow y)$;
- (iv) $w \leftrightarrow z \geq d^n(x \leftrightarrow y)$, *for some* $n \in \omega$.

Proof. Part (3) follows easily from the first two parts. To prove part (1), let $x \in A$ and let $F' = {\uparrow}\{d^n x \mid n \in \omega\}$. By definition, if F is a congruence-filter and $x \in F$, then we have $F' \subseteq F$. It remains to show that F' is a congruence-filter. Because d is descending, it follows that $F' = \bigcup\{{\uparrow}d^n x \mid n \in \omega\}$, which is a directed union of filters. So F' is a filter, and it remains to check that it is closed under d. But that follows because d is order-preserving.

For part (2), let $\alpha \in \mathrm{Con}(\mathbf{A})$ and let $(x, y) \in \alpha$. Then

$$\begin{aligned} (x,y) \in \alpha &\implies (x \leftrightarrow y, 1) \in \alpha \\ &\implies x \leftrightarrow y \in 1/\alpha \\ &\implies \mathrm{Fg}^{\mathbf{A}}(x \leftrightarrow y) \subseteq 1/\alpha \\ &\implies \theta(\mathrm{Fg}^{\mathbf{A}}(x \leftrightarrow y)) \subseteq \theta(1/\alpha) = \alpha. \end{aligned}$$

Clearly $(x, y) \in \theta(\mathrm{Fg}^{\mathbf{A}}(x \leftrightarrow y))$, so $\theta(\mathrm{Fg}^{\mathbf{A}}(x \leftrightarrow y))$ is the smallest congruence containing x and y. □

Theorem 5.3.3. *Let* $\mathbf{A}$ *be an EHA and assume that* d *is a descending congruence-filter term on* $\mathbf{A}$*.*

(1) $\mathbf{A}$ *is subdirectly irreducible if and only if there exists* $b \in A\backslash\{1\}$ *such that, for all* $x \in A\backslash\{1\}$*, there is some* $n \in \omega$ *such that* $b \geq d^n x$*.*

(2) $\mathbf{A}$ *is simple if and only if, for all* $x \in A\backslash\{1\}$*, there exists* $n \in \omega$ *such that* $d^n x = 0$*.*

Proof. Assume that $\mathbf{A}$ is subdirectly irreducible. Then the monolith is equal to $\mathrm{Cg}^{\mathbf{A}}(b, 1)$, for some $b \in A$. Since $1/\,\mathrm{Cg}^{\mathbf{A}}(b, 1) = \mathrm{Fg}^{\mathbf{A}}(b)$, we must have that $\mathrm{Fg}^{\mathbf{A}}(b) \subseteq \mathrm{Fg}^{\mathbf{A}}(x)$, for all $x \in A/\{1\}$, which proves one direction of the first part. For the converse, assume that b is as stated. Then $\mathrm{Fg}^{\mathbf{A}}(b)$ is contained in every non-trivial congruence-filter of $\mathbf{A}$, so $\theta(\mathrm{Fg}^{\mathbf{A}}(b))$ is the monolith of $\mathbf{A}$. For the second part, simply note that $\mathbf{A}$ is simple if and only if the monolith and $\mathrm{Cg}^{\mathbf{A}}(0, 1)$ are equal. □

Theorem 5.3.4. *Let* $\mathbf{A}$ *be an EHA and assume that* d *is a descending congruence-filter term on* $\mathbf{A}$*. Then* $\mathbf{A}$ *has the congruence extension property.*

Proof. Let $\mathbf{B}$ be a subalgebra of $\mathbf{A}$ and let F be a congruence-filter of $\mathbf{B}$. Then define

$$F' = \{z \in A \mid (\exists n \in \omega)(\exists x_1, \ldots, x_k \in F)\ z \geq d^n(x_1 \wedge \ldots \wedge x_k)\}.$$

Notice that this is just the congruence-filter of $\mathbf{A}$ generated by F. We claim that $F' \cap B = F$. Clearly $F \subseteq F' \cap B$, so let $x \in F' \cap B$. Then there exists $n \in \omega$ and $x_1, \ldots, x_k \in F$ such that $x \geq d^n(x_1 \wedge \ldots \wedge x_k) \in F$. Since $x \in B$, we then have $x \in F$, and we are done. □

Observe that these three results are substantial generalisations of the results in Section 1.4 and Section 1.5 by Köhler, Beazer and Sankappanavar.

5.4 Equationally definable principal congruences

Definition 5.4.1. Let $\mathcal{V}$ be a variety of any signature. Then $\mathcal{V}$ *has definable principal congruences* (DPC) if there exists a first-order formula $\varphi(x, y, z, w)$ in the language of $\mathcal{V}$ such that, for all $\mathbf{A} \in \mathcal{V}$ and all $a, b, c, d \in A$, we have $(a, b) \in \mathrm{Cg}^{\mathbf{A}}(c, d)$ if and only if $\mathbf{A} \models \varphi(a, b, c, d)$. We then say that φ *defines principal congruences on* $\mathcal{V}$. If there exists a finite conjunction of equations that defines principal congruences on $\mathcal{V}$, then we say $\mathcal{V}$ has *equationally definable principal congruences* (EDPC).

Varieties with equationally definable principal congruences were studied extensively in a series of papers by Blok, Köhler, and Pigozzi [9–12, 58]. Provided that the variety has a congruence-filter term, EDPC has an easy characterisation.

Theorem 5.4.2. *Let* $\mathcal{V}$ *be a variety of EHAs and assume* $\mathcal{V}$ *has a descending congruence-filter term* d. *The following are equivalent*:

(1) $\mathcal{V}$ *has EDPC*;

(2) $\mathcal{V}$ *has DPC*;

(3) *there exists* $n \in \omega$ *such that* $\mathcal{V} \models d^{n+1}x = d^n x$.

Proof. Clearly (1) implies (2). To prove that (2) implies (3), assume $\mathcal{V}$ has DPC and suppose, by way of contradiction, that for all $i \in \omega$, there exists $\mathbf{A}_i \in \mathcal{V}$ and $a_i \in A_i$ such that $d^i a \neq d^{i+1} a_i$. Let U be a non-principal ultrafilter on ω and consider the ultraproduct $\mathbf{A} = \prod_{i \in \omega} A_i/U$. Let φ be a formula that defines principal congruences on $\mathcal{V}$, let $a = \langle a_i \mid i \in \omega\rangle/U$, and let $b = \langle d^{i+1} a_i \mid i \in \omega\rangle/U$. For each coordinate i, we have $(d^{i+1}a_i, 1_i) \in \mathrm{Cg}^{\mathbf{A}_i}(a_i, 1_i)$ by Theorem 5.3.2, and so $\mathbf{A}_i \models \varphi(d^{i+1}a_i, 1_i, a_i, 1_i)$. It follows that $\mathbf{A} \models \varphi(b, 1, a, 1)$. So $(b, 1) \in \mathrm{Cg}^{\mathbf{A}}(a, 1)$, and hence, by Theorem 5.3.2, there exists $k \in \omega$ such that $b \geq d^k a$. Then by the properties of ultraproducts, there exists a cofinite $I \subseteq \omega$ such that $d^{i+1}a_i \geq d^k a_i$, for all $i \in I$. We must have some $j \geq k$ such that $d^{j+1}a_j \geq d^k a_j$, as otherwise I is finite. But then, since d is descending, we have $d^k a_j \geq d^j a_j \geq d^{j+1} a_j \geq d^k a_j$, and it follows that $d^j a_j = d^{j+1} a_j$, a contradiction. Finally, to see that (3) implies (1), if there exists $n \in \omega$ such that $\mathcal{V} \models d^n x = d^{n+1} x$, then by Theorem 5.3.2, the equation given by

$$a \leftrightarrow b \geq d^n(c \leftrightarrow d)$$

defines principal congruences on $\mathcal{V}$. □

5.5 Splitting algebras

We began investigating splittings in Section 3.2, and in Section 3.3 we proved that every finite subdirectly irreducible algebra in a locally finite congruence-distributive variety is a splitting algebra. In this section we continue the investigation of splitting algebras in certain classes of EHAs. To be more precise, we will characterise finite non-splitting algebras. If the variety is generated by its finite members, then by Lemma 3.2.10, we have characterised all of the splitting algebras.

Even in that case, the characterisation on its own gives no insight into the number of splitting algebras. To prove that an algebra is not splitting, the result demands the existence of a countably infinite number of algebras satisfying certain properties. To pin down more precisely the splitting algebras in a given variety, a construction specific to that variety is needed. In Chapter 8, we will combine the results of this section with results of Chapter 2 to prove that, up to isomorphism, there are exactly two splitting algebras in the varieties of H^+-algebras and double Heyting algebras. Recall that $\mathcal{L}(\mathcal{V})$ denotes the lattice of subvarieties of a variety $\mathcal{V}$ and refer to Definition 3.2.9 for the definition of a splitting algebra.

For the remainder of this section, let $\mathcal{V}$ be a fixed but arbitrary variety of EHAs and assume

(1) $\mathcal{V}$ has a finite signature $F = \langle M, \vee, \wedge, \rightarrow, 0, 1\rangle$,

(2) $\mathcal{V}$ has a descending congruence-filter term d.

The proof in this section is due to Tomasz Kowalski, and is a modification of an argument by Kowalski and Ono [63]. It is included for completeness, and a slightly more general approach is included in a manuscript currently under preperation by Davey, Kowalski, and the author.

Definition 5.5.1. Let $\mathbf{A}$ be a finite subdirectly irreducible algebra in $\mathcal{V}$, let $n = |A|$, and let $A = \{a_1, \dots, a_n\}$. The *term-diagram* of $\mathbf{A}$ is the n-ary term with variables $\{x_{a_1}, \dots, x_{a_n}\}$ given by

$$\Delta^{\mathcal{V}}_{\mathbf{A}}(x_{a_1}, \dots, x_{a_n}) = \bigwedge\{x_{f(a_1,\dots,a_k)} \leftrightarrow f(x_{a_1}, \dots, x_{a_k}) \mid a_1, \dots, a_k \in A \text{ and } f \in F\}.$$

If $\mathcal{V}$ is clear, we will just write $\Delta_{\mathbf{A}}$ instead. Let $\mu_{\mathbf{A}}$ denote the monolith of $\mathbf{A}$. Since $\mathbf{A}$ is finite, the filter $1/\mu_{\mathbf{A}}$ has a minimum element. Denote the minimum element of $1/\mu_{\mathbf{A}}$ by $\eta_{\mathbf{A}}$. If $\mathbf{A}$ is clear from context, then the subscript may be omitted.

Note that $\eta_{\mathbf{A}} \neq 1$, because otherwise $\mu_{\mathbf{A}} = 0_{\mathbf{A}}$. As an example, if $\mathbf{A} \in \mathcal{H}^+$, then

$\Delta_{\mathbf{A}}^{\mathcal{H}^+}$ is given by

$$\begin{aligned}\Delta_{\mathbf{A}}^{\mathcal{H}^+}(\overline{x}) = \bigwedge\{[x_{a\wedge b} \leftrightarrow (x_a \wedge x_b)] \wedge [x_{a\vee b} \leftrightarrow (x_a \vee x_b)] \\ \wedge [x_{a\to b} \leftrightarrow (x_a \to x_b)] \wedge [x_{\sim a} \leftrightarrow {\sim} x_a] \\ \wedge [x_0 \leftrightarrow 0] \wedge [x_1 \leftrightarrow 1] \mid a, b \in A\},\end{aligned}$$

and $\eta_{\mathbf{A}} = 0$. In the general case, η need not be a constant in the language of $\mathcal{V}$.

Lemma 5.5.2. *Let $\mathbf{A}, \mathbf{B} \in \mathcal{V}$. Assume that $\mathbf{A}$ is finite and subdirectly irreducible. Then $\mathbf{A} \leq \mathbf{B}$ if and only if, for each $a \in A$, there exists $b_a \in B$ such that, under the assignment $x_a \mapsto b_a$, we have $\Delta_{\mathbf{A}}^{\mathcal{V}}(\overline{x}) = 1$ and $x_{\eta_{\mathbf{A}}} \neq 1$.*

Proof. For the forward direction, without loss of generality, assume that $\mathbf{A}$ is a subalgebra of $\mathbf{B}$. For each $a \in A$, let $b_a = a$, and assign the variables as in the statement of the theorem. Then $x_{\eta_{\mathbf{A}}} = b_{\eta_{\mathbf{A}}} = \eta_{\mathbf{A}} \neq 1$. Also, for each $f \in F$, if n is the arity of f and $a_1, \dots, a_n \in A$, then

$$\begin{aligned}x_{f(a_1,\dots,a_n)} \leftrightarrow f(x_{a_1}, \dots, x_{a_n}) &= b_{f(a_1,\dots,a_n)} \leftrightarrow f(b_{a_1}, \dots, b_{a_n}) \\ &= f(a_1, \dots, a_n) \leftrightarrow f(a_1, \dots, a_n) = 1,\end{aligned}$$

and it follows that $\Delta_{\mathbf{A}}(\overline{x}) = 1$. For the converse, for each $a \in A$, let $b_a \in B$ and assume b_a has the stated properties. Since $\Delta_{\mathbf{A}}(\overline{x}) = 1$, we must have

$$b_{f(a_1,\dots,a_k)} \leftrightarrow f(b_{a_1}, \dots, b_{a_k}) = 1,$$

for all $f \in F$ and all $a_1, \dots, a_k \in A$. Hence $b_{f(a_1,\dots,a_n)} = f(b_{a_1}, \dots, b_{a_k})$, which says exactly that $b\colon a \mapsto b_a$ is a homomorphism from $\mathbf{A}$ to $\mathbf{B}$. Moreover, since $b_1 = 1 \neq b_{\eta_{\mathbf{A}}}$, the kernel of b does not contain $(1, \eta_{\mathbf{A}})$. Since $(1, \eta_{\mathbf{A}})$ generates the monolith of $\mathbf{A}$, it follows that the kernel of b is the trivial congruence on $\mathbf{A}$. Hence b is an embedding. □

Lemma 5.5.3. *Let $\mathbf{A} \in \mathcal{V}$ and assume $\mathbf{A}$ is finite and subdirectly ireducible. The following are equivalent:*

(1) *$\mathbf{A}$ is not a splitting algebra in $\mathcal{V}$;*

(2) $(\forall i \in \mathbb{N})(\exists \mathbf{B} \in \mathcal{V})$ $\mathbf{A} \notin \mathrm{Var}(\mathbf{B})$ *and* $\mathbf{B} \not\models d^i \Delta_{\mathbf{A}}^{\mathcal{V}}(\overline{x}) \leq x_{\eta_{\mathbf{A}}}$.

Proof. First assume that (2) holds. For each $i \in \mathbb{N}$, choose an algebra $\mathbf{B}_i$ in $\mathcal{V}$ such that $\mathbf{A} \notin \mathrm{Var}(\mathbf{B}_i)$ and $\mathbf{B}_i \not\models d^i \Delta_{\mathbf{A}}(\overline{x}) \leq x_\eta$. Let $k = |A|$, and for each $i \in \mathbb{N}$, choose a sequence of elements $b_i = \langle b_i(1), \dots, b_i(k-1), s_i\rangle$ from $\mathbf{B}_i$ such that $d^i \Delta_{\mathbf{A}}(b_i) \nleq s_i$. Let $\mathbf{B} = \prod_{i\in\mathbb{N}} \mathbf{B}_i$, let

$$\overline{b} = \big\langle \langle b_i(1) \mid i \in \mathbb{N}\rangle, \dots, \langle b_i(k-1) \mid i \in \mathbb{N}\rangle, \langle s_i \mid i \in \mathbb{N}\rangle \big\rangle,$$

and let $s = \langle s_i \mid i \in \mathbb{N} \rangle$. By construction, we have $d^i\Delta_{\mathbf{A}}(\bar{b}) \nleq s$, for all $i \in \mathbb{N}$, so $s \notin \mathrm{Fg}^{\mathbf{B}}(\Delta_{\mathbf{A}}(\bar{b}))$. Let $\theta = \theta(\mathrm{Fg}^{\mathbf{B}}(\Delta_{\mathbf{A}}(\bar{b})))$. Then $\Delta_{\mathbf{A}}(\bar{b})/\theta = 1/\theta$ and $s/\theta \neq 1/\theta$, which implies $\mathbf{A} \leq \mathbf{B}/\theta$ by Lemma 5.5.2. So $\mathbf{A} \in \mathrm{Var}(\mathbf{B})$. Now suppose, by way of contradiction, that $\mathbf{A}$ is a splitting algebra. Then there is a largest subvariety $\mathcal{B}$ of $\mathcal{V}$ such that $\mathbf{A} \notin \mathcal{B}$, and by assumption we have $\mathbf{B}_i \in \mathcal{B}$, for each $i \in \mathbb{N}$. But then $\mathbf{B} \in \mathcal{B}$, implying $\mathbf{A} \in \mathcal{B}$, which is a contradiction.

To show that (1) implies (2) we will prove the contrapositive. Assume that there exists $i \in \mathbb{N}$ such that, for all $\mathbf{B} \in \mathcal{V}$,

$$\mathbf{A} \notin \mathrm{Var}(\mathbf{B}) \implies \mathbf{B} \models d^i\Delta_{\mathbf{A}}(\overline{x}) \leq x_{\eta_{\mathbf{A}}}.$$

Let m be the smallest such i and let $\mathcal{B}$ be the subvariety of $\mathcal{V}$ defined by the inequality $d^m\Delta_{\mathbf{A}}(\overline{x}) \leq x_{\eta_{\mathbf{A}}}$. We claim that $(\mathrm{Var}(\mathbf{A}), \mathcal{B})$ is a splitting pair. Certainly $\mathbf{A} \notin \mathcal{B}$, because under the assignment $x_a \mapsto a$ we have $d^m\Delta_{\mathbf{A}}(\overline{x}) = 1 \nleq \eta_{\mathbf{A}}$. Now let $\mathcal{A}$ be a subvariety of $\mathcal{V}$ with $\mathbf{A} \notin \mathcal{A}$ and let $\mathbf{F}$ be the free algebra in $\mathcal{A}$ on countably many generators. Because $\mathbf{A} \notin \mathcal{A}$, it follows by assumption that $\mathbf{F} \models d^i\Delta_{\mathbf{A}}(\overline{x}) \leq x_{\eta_{\mathbf{A}}}$, so $\mathcal{A}$ satisfies the same inequality. Hence $\mathcal{A} \subseteq \mathcal{B}$, completing the proof. $\square$

6

Expansions of H^+-algebras

In this chapter, we restrict our attention to expansions of dually pseudocomplemented Heyting algebras. Naturally, this includes double Heyting algebras as well. The symmetry of double Heyting algebras means that, for congruences, the choice between filters and ideals is completely arbitrary. This connection was noted by Köhler [59], who gave a bijection between congruence-filters and congruence-ideals of double Heyting algebras, but without mention of the lattice structure. It is easy to show that the bijection is a lattice isomorphism. It turns out that, when combined with the Heyting algebra operations, the dual pseudocomplement is strong enough to ensure that congruences are also determined by ideals. We will prove that, for any algebra with an H^+-algebra reduct, there is an isomorphism between the lattice of its congruence-filters and the lattice of its congruence-ideals. The isomorphism leads into a sort of conjugacy result for order-preserving unary functions, which lets us turn "congruence-ideal terms" for H^+-algebras into congruence-filter terms. This allows us to show that congruence-filter terms exist for certain expansions of double Heyting algebras not already considered, because the dual implication in the signature permits Hasimoto's normalisation technique to dualise. We have not been able to analogise it for H^+-algebras.

In the remainder of this chapter, we focus on the connection between discriminator varieties and semisimple varieties. In the presence of the dual pseudocomplement and a congruence-filter term, discriminator varieties and semisimple varieties are one and the same. Our proof of this is based on a similar result by Kowalski and Kracht for Boolean algebras with operators [62], and on a similar result by the author [85], applicable exclusively to double Heyting algebras and H^+-algebras. Both of these results follow from the argument in this chapter. The content of this chapter is based on the author's paper in *Studia Logica* [86], and is entirely the original work of the author. We note that the results in the first section are stronger than the related results in [86]. Specifically, the author proved in [86] only that the isomorphism mentioned earlier exists for H^+-algebras, but not for their expansions.

6.1 Conjugacy between filters and ideals

Definition 6.1.1. Let $\mathbf{A}$ be an EHA and let I be an ideal of $\mathbf{A}$. Define the binary relation $\lambda(I)$ by

$$\lambda(I) = \{(x, y) \in A^2 \mid (\exists z \in I)\ x \vee z = y \vee z\}.$$

In general, $\lambda(I)$ is a congruence for every distributive lattice, but it need not be a Heyting algebra congruence. We will say that an ideal I is a *congruence-ideal* (*of* $\mathbf{A}$) if $\lambda(I)$ is a congruence on $\mathbf{A}$. Let $\mathrm{Idl}(\mathbf{A})$ denote the set of congruence-ideals of $\mathbf{A}$. If $\mathrm{Idl}(\mathbf{A})$ is a lattice, then we will denote that lattice by $\mathbf{Idl}(\mathbf{A})$.

It is easily verified that $\mathrm{Idl}(\mathbf{A})$ is closed under finite intersections, but there is no reason for $\mathrm{Idl}(\mathbf{A})$ to be a lattice in general. On the other hand, if $\mathbf{A}$ has a dual Heyting algebra reduct, then by the dual of Lemma 1.2.8, every congruence is of the form $\lambda(I)$, for some ideal I. So $\mathrm{Idl}(\mathbf{A})$ is a lattice in that case. Characterising these ideals for double Heyting algebras is just a matter of dualising Theorem 1.5.6.

Definition 6.1.2. Let $\mathbf{A}$ be an EHA. If there is a term $\sim$ in the language of $\mathbf{A}$ such that $\sim$ is the dual pseudocomplement on $\mathbf{A}$, we then say that $\mathbf{A}$ is a *dually pseudocomplemented EHA*. A class $\mathcal{K}$ of EHAs of common signature is *dually pseudocomplemented* if there is a term $\sim$ in the language of $\mathcal{K}$ such that $\sim$ is the dual pseudocomplement on every algebra in $\mathcal{K}$. Assume that $\mathbf{A}$ is a dually pseudocomplemented EHA. For every filter F of $\mathbf{A}$, let $\mathcal{I}(F)$ be the set defined by

$$\mathcal{I}(F) = {\downarrow}{\sim}F = \{y \in A \mid (\exists x \in F)\ y \leq {\sim}x\}.$$

Similarly, for each ideal I of $\mathbf{A}$, let $\mathcal{F}(I) = {\uparrow}\neg I$. A filter is called a *normal filter* if it is closed under $\neg\sim$, and dually, an ideal is called a *normal ideal* if it is closed under $\sim\neg$.

The set of normal filters and the set of normal ideals of a dually pseudocomplemented EHA are easily shown to be complete lattices. Notice that they are exactly the congruence-filters and congruence-ideals of double Heyting algebras.

Lemma 6.1.3. *Let $\mathbf{A}$ be a dually pseudocomplemented EHA, let F be a filter of $\mathbf{A}$, and let I be an ideal of $\mathbf{A}$.*

(1) *$\mathcal{I}(F)$ is an ideal of $\mathbf{A}$, and $\mathcal{F}(I)$ is a filter of $\mathbf{A}$.*

(2) *If F is a normal filter, then $\mathcal{I}(F)$ is a normal ideal.*

(3) *If I is a normal ideal, then $\mathcal{F}(I)$ is a normal filter.*

Proof. To see that $\mathcal{I}(F)$ is an ideal, first, it is clear that it is a non-empty downset. Now let $x, y \in \mathcal{I}(F)$. Then there exists $x', y' \in F$ such that $x \leq {\sim}x'$ and $y \leq {\sim}y'$. Then $x \vee y \leq {\sim}x' \vee {\sim}y' = {\sim}(x' \wedge y')$, and since $x' \wedge y' \in F$, it follows that $x \vee y \in \mathcal{I}(F)$. So $\mathcal{I}(F)$ is an ideal. A dual argument shows that $\mathcal{F}(I)$ is a filter. For part (2), assume F is closed under $\neg{\sim}$ and let $x \in \mathcal{I}(F)$. Then there exists $x' \in F$ such that $x \leq {\sim}x'$, so $\neg x \geq \neg{\sim}x' \in F$. It then follows that ${\sim}\neg x \leq {\sim}\neg{\sim}x' \in {\sim}F \subseteq \mathcal{I}(F)$. Part (3) is proved dually. □

Theorem 6.1.4. *Let* $\mathbf{A}$ *be a dually pseudocomplemented EHA. The maps* $\mathcal{I}$ *and* $\mathcal{F}$ *are mutually inverse isomorphisms between the lattice of normal filters of* $\mathbf{A}$ *and the lattice of normal ideals of* $\mathbf{A}$*.*

Proof. It follows from Lemma 6.1.3 that the two maps are well-defined, and it is easy to see that they are order-preserving. To show that they are inverse with one another, we will first show that $\mathcal{I}(\mathcal{F}(I)) = I$. Let $x \in I$. Since $x \leq {\sim}\neg x$, we have $x \in {\downarrow}{\sim}{\uparrow}\neg I = \mathcal{I}(\mathcal{F}(I))$. Conversely, if $x \in \mathcal{I}(\mathcal{F}(I))$, then there exists $x' \in {\uparrow}\neg I$ such that $x \leq {\sim}x'$. Then there exists $x'' \in I$ such that $x' \geq \neg x''$. Because $\sim$ is order-reversing, we then have ${\sim}x' \leq {\sim}\neg x''$, and since I is closed under ${\sim}\neg$, it follows that ${\sim}\neg x'' \in I$. Thus, ${\sim}x' \in I$, and so $x \in I$. A dual argument shows that $\mathcal{F}(\mathcal{I}(F)) = F$. □

Corollary 6.1.5. *Let* $\mathbf{A}$ *be an* H^+*-algebra. Then the lattice of normal ideals of* $\mathbf{A}$ *is isomorphic to* $\mathbf{Con}(\mathbf{A})$*.*

This does not yet show that if I is a normal filter of an H$^+$-algebra, then $\lambda(I)$ is a congruence. But it does show that $\theta(\mathcal{F}(I))$ is a congruence. We will show soon that $\theta \circ \mathcal{F} = \lambda$ when $\mathcal{F}$ and λ are restricted to normal ideals.

Lemma 6.1.6. *Let* $\mathbf{A}$ *be a dually pseudocomplemented EHA, let* F *be a normal filter of* $\mathbf{A}$*, and let* $x, y \in A$*.*

(1) $x \vee {\sim}(x \leftrightarrow y) = y \vee {\sim}(x \leftrightarrow y)$.

(2) $x \leftrightarrow y \in F$ *if and only if* ${\sim}(x \leftrightarrow y) \in \mathcal{I}(F)$.

Proof. For (1), we have $((x \leftrightarrow y) \wedge x) \vee {\sim}(x \leftrightarrow y) = x \vee {\sim}(x \leftrightarrow y)$, and since $(x \leftrightarrow y) \wedge x = (x \leftrightarrow y) \wedge y$, it follows that $x \vee {\sim}(x \leftrightarrow y) = y \vee {\sim}(x \leftrightarrow y)$. For (2), if $x \leftrightarrow y \in F$, then by definition, we have ${\sim}(x \leftrightarrow y) \in \mathcal{I}(F)$. Conversely, if ${\sim}(x \leftrightarrow y) \in \mathcal{I}(F)$, then $\neg{\sim}(x \leftrightarrow y) \in \mathcal{F}(\mathcal{I}(F)) = F$, and because $\neg{\sim}x \leq x$, it follows that $x \leftrightarrow y \in F$. □

Recall that we let M denote the set of additional operations of an EHA.

Definition 6.1.7. Let $\mathbf{A}$ be an EHA. If $\dot{-} \in M$, then we will say that $\mathbf{A}$ is an *expanded double Heyting algebra* (or EDHA for short). For all $x, y \in A$, let $x \div y$ be an abbreviation for $(x \dot{-} y) \vee (y \dot{-} x)$.

By the dual of Lemma 1.2.7, if $\mathbf{A}$ is an EDHA and $x, y \in A$, then $x \div y$ is the smallest $z \in A$ such that $x \vee z = y \vee z$. Consequently, by Lemma 6.1.6, if $\mathbf{A}$ is an EDHA, then $\sim(x \leftrightarrow y) \geq x \div y$.

Proposition 6.1.8. *Let* $\mathbf{A}$ *be an* EDHA *and let* F *be a normal filter of* $\mathbf{A}$*. The following are equivalent:*

(1) $x \leftrightarrow y \in F$;

(2) $\sim(x \leftrightarrow y) \in \mathcal{I}(F)$;

(3) $x \div y \in \mathcal{I}(F)$.

Proof. Lemma 6.1.6 proves the equivalence of (1) and (2). The implication from (2) to (3) holds because $\sim(x \leftrightarrow y) \geq x \div y$. All that remains is to see that (3) implies (2). By the dual of Theorem 1.5.6 and because $\mathcal{I}(F)$ is a normal filter, it follows that $\lambda(\mathcal{I}(F))$ is a double Heyting algebra congruence. Thus,

$$\begin{aligned} x \div y \in \mathcal{I}(F) &\implies (x, y) \in \lambda(\mathcal{I}(F)) \\ &\implies (x \leftrightarrow y, 1) \in \lambda(\mathcal{I}(F)) \\ &\implies (\sim(x \leftrightarrow y), 0) \in \lambda(\mathcal{I}(F)) \\ &\implies \sim(x \leftrightarrow y) \in 0/\lambda(\mathcal{I}(F)) = \mathcal{I}(F), \end{aligned}$$

as claimed. □

Because $\div$ is not defined for H$^+$-algebras, the dual of Theorem 1.2.8 does not apply to dually pseudocomplemented EHAs. We have a similar construction instead.

Proposition 6.1.9. *Let* $\mathbf{A}$ *be a dually pseudocomplemented* EHA *and let* I *be a normal ideal of* $\mathbf{A}$*. Then*

$$\lambda(I) = \{(x, y) \in A^2 \mid \sim(x \leftrightarrow y) \in I\}.$$

Proof. By Lemma 6.1.6 and by definition of λ, it follows that if $\sim(x \leftrightarrow y) \in I$, then $(x, y) \in \lambda(I)$. Conversely, assume there exists $z \in I$ such that $x \vee z = y \vee z$. Then $x \wedge \neg z = y \wedge \neg z$, so $x \leftrightarrow y \geq \neg z$ by Lemma 1.2.7. Since $z \in I$, we have $\neg z \in \mathcal{F}(I)$, and thus $x \leftrightarrow y \in \mathcal{F}(I)$. Hence $\sim(x \leftrightarrow y) \in \mathcal{I}(\mathcal{F}(I)) = I$. □

Definition 6.1.10. Let $\mathbf{A}$ be a dually pseudocomplemented EHA and let f be an n-ary operation on A. We say that a normal ideal I is *compatible with* f if the following implication is satisfied, for all $x_1, y_1, \dots, x_n, y_n \in A$:

$$\sim(x_1 \leftrightarrow y_1), \dots, \sim(x_n \leftrightarrow y_n) \in I \implies \sim(f(x_1, \dots, x_n) \leftrightarrow f(y_1, \dots, y_n)) \in I.$$

The next result follows from Lemma 6.1.6.

Proposition 6.1.11. *Let* $\mathbf{A}$ *be a dually pseudocomplemented* EHA*, let* F *be a normal filter of* $\mathbf{A}$*, and let* f *be an operation on* A*. Then* F *is compatible with* f *if and only if* $\mathcal{I}(F)$ *is compatible with* f*.*

Theorem 6.1.12. *Let* $\mathbf{A}$ *be a dually pseudocomplemented* EHA *and let* I *be an ideal of* $\mathbf{A}$*.*

(1) *If* I *is normal, then* $\lambda(I) = \theta(\mathcal{F}(I))$*.*

(2) *The following are equivalent:*

  (i) I *is a congruence-ideal;*

  (ii) I *is normal and* $\mathcal{F}(I)$ *is a congruence-filter;*

  (iii) I *is normal and compatible with* M*.*

(3) $\mathrm{Idl}(\mathbf{A})$ *is a lattice and* λ *is an isomorphism from* $\mathbf{Idl}(\mathbf{A})$ *to* $\mathbf{Con}(\mathbf{A})$*.*

Proof. By Proposition 6.1.9, if I is normal, then

$$\lambda(I) = \{(x, y) \mid \sim(x \leftrightarrow y) \in I\}.$$

By Lemma 6.1.6, this equals $\{(x, y) \mid x \leftrightarrow y \in \mathcal{F}(I)\}$, which is just $\theta(\mathcal{F}(I))$. This proves part (1). Now for part (2), assume that $\lambda(I)$ is a congruence on $\mathbf{A}$. If $x \in I$, then $(x, 0) \in \lambda(I)$ by definition, and because $\lambda(I)$ is a congruence, we have $(\sim\neg x, 0) \in \lambda(I)$. This implies $\sim\neg x \in I$, so I is normal. Then (2ii) holds by part (1). Part (2ii) implies (2i) by part (1) as well. Parts (2ii) and (2iii) are equivalent by Theorem 5.1.2 and Proposition 6.1.11. Part (2) now ensures that the map $\mathcal{F}\colon \mathrm{Idl}(\mathbf{A}) \to \mathrm{Fil}(\mathbf{A})$ given by $I \mapsto \mathcal{F}(I)$ is well defined. The last part then holds by part (1), because θ and $\mathcal{F}$ are order-isomorphisms. □

This suggests an analogous approach to congruences using "congruence-ideal terms" instead of congruence-filter terms, provided that the dual pseudocomplement is present. The next lemma tells us we would just be looking at congruence-filter terms in disguise.

Lemma 6.1.13. *Let* $\mathbf{A}$ *be a dually pseudocomplemented* EHA*, let* F *be a normal filter of* $\mathbf{A}$*, and let* I *be a normal ideal of* $\mathbf{A}$*. Let* f *be an order-preserving unary operation on* A*.*

(1) *I is closed under f if and only if $\mathcal{F}(I)$ is closed under $\neg f{\sim}$.*

(2) *F is closed under f if and only if $\mathcal{I}(F)$ is closed under ${\sim} f\neg$.*

Proof. For part (1), first assume that $\mathcal{F}(I)$ is closed under $\neg f{\sim}$ and let $x \in I$. By Theorem 6.1.4, we have that $I = \mathcal{I}(\mathcal{F}(I))$. So there exists $x' \in \mathcal{F}(I)$ such that $x \leq {\sim}x'$. Since $\mathcal{F}(I)$ is closed under $\neg f{\sim}$, we have $\neg f{\sim}x' \in \mathcal{F}(I)$, and so ${\sim}\neg f{\sim}x' \in \mathcal{I}(\mathcal{F}(I)) = I$. Since f is order-preserving, we have

$$fx \leq f{\sim}x' \leq {\sim}\neg f{\sim}x' \in I$$

Thus, $fx \in I$. Conversely, assume I is closed under f and let $x \in \mathcal{F}(I)$. Then there exists $x' \in I$ such that $x \geq \neg x'$, implying ${\sim}x \leq {\sim}\neg x'$. Then, since f is order-preserving, we have $f{\sim}x \leq f{\sim}\neg x'$. Since I is a normal ideal, it is closed under ${\sim}\neg$, and so ${\sim}\neg x' \in I$, implying $f{\sim}\neg x' \in I$. Hence $f{\sim}x \in I$, and it follows that $\neg f{\sim}x \in \mathcal{F}(I)$. Part (2) follows by a dual argument. □

We can freely dualise Hasimoto's normalisation technique when $\dot{-}$ is in the signature. On the other hand, the term ${\sim}(x \leftrightarrow y)$ does not share enough properties with $x \div y$ to utilise the technique verbatim for dually pseudocomplemented EHAs.

Definition 6.1.14. Let $\mathbf{A}$ be a double Heyting algebra and let $f\colon A^n \to A$ be a map. Let $\mathbf{A}^\partial$ denote the (order-theoretic) dual of $\mathbf{A}$. We say that f is a *dual operator* if it is an operator on $\mathbf{A}^\partial$, and we say that f is *dually normal* if it is normal on $\mathbf{A}^\partial$. Similarly, we say that f is a *dual anti-operator* if it is an anti-operator on $\mathbf{A}^\partial$, and it is *dually anti-normal* if it is anti-normal on $\mathbf{A}^\partial$. For convenience, if f is a dually normal dual operator, then we will say that f is a *dual normal operator*. Similarly, if f is a dually anti-normal dual anti-operator, then we will say that f is a *dual anti-normal operator*. We say that f is a *generalised operator* if f is a normal operator, an anti-normal operator, a dual normal operator, or an anti-normal dual operator. For each $a \in A$ and each $k \leq n$, let $\bar{f}^{(k)}a$ be an abbreviation for $f_k(a)(1, \ldots, 1)$. That is,

$$\bar{f}^{(k)}a = f(1, \ldots, 1, a, 1, \ldots, 1),$$

where a is in the k-th position.

Theorem 6.1.15. *Let* $\mathbf{A}$ *be an* EDHA *and let* f *be an operation in the signature of* $\mathbf{A}$*. Assume that* f *is a generalised operator and let* $\hat{t}$ *denote the term defined by*

$$\hat{t}x = \begin{cases} \bigwedge_{k\le n} f^{(k)}x, & \text{if } f \text{ is normal,} \\ \bigwedge_{k\le n} \neg f^{(k)}x, & \text{else if } f \text{ is anti-normal,} \\ \bigwedge_{k\le n} \neg \bar{f}^{(k)}{\sim}x, & \text{else if } f \text{ is dually normal,} \\ \bigwedge_{k\le n} \bar{f}^{(k)}{\sim}x, & \text{else if } f \text{ is dually anti-normal.} \end{cases}$$

Then the term t *given by* $tx = \neg{\sim}x \wedge \hat{t}x$ *is a compatibility term for the set* $\{\dot{-}, f\}$*.*

Proof. Let F be a filter of $\mathbf{A}$. We want to show that F is closed under t if and only if F is compatible with both $\dot{-}$ and f. If F is closed under t, then F is a normal filter and closed under $\hat{t}$. Also, if F is compatible with $\{\dot{-}, f\}$, then F is a normal filter and compatible with f. Thus, it suffices to assume that F is a normal filter and prove that F is closed under $\hat{t}$ if and only if F is compatible with f.

The first two cases are from Corollary 5.2.15. For the remaining cases, first assume f is a dually normal dual operator and that F is compatible with f. By Proposition 6.1.11, the ideal $I := \mathcal{I}(F)$ is compatible with f. By the dual of Corollary 5.2.15, it follows that I is compatible with f if and only if I is closed under $sx := \bigvee_{k\le n} \bar{f}^{(k)}x$. Hence, by Lemma 6.1.13, the filter F is closed under $\neg s{\sim}x$, which in turn is equal to $\bigwedge_{k\le n} \neg\bar{f}^{(k)}{\sim}x$. Conversely, if F is closed under $\neg s{\sim}x$, then I is closed under s. It follows that I is compatible with f, and so F is compatible with f by Proposition 6.1.11.

A similar argument proves that if f is a dual anti-normal operator, then a normal ideal I is compatible with f if and only if I is closed under $\bigvee_{k\le n} {\sim}\bar{f}^{(k)}x$. So F is compatible with f if and only if F is closed under $\neg \bigvee_{k\le n} {\sim}\bar{f}^{(k)}{\sim}x$, and that is equal to $\neg{\sim} \bigwedge_{k\le n} \bar{f}^{(k)}{\sim}x$. Because F is normal and $\neg{\sim} \bigwedge_{k\le n} \bar{f}^{(k)}{\sim}x \le \bigwedge_{k\le n} \bar{f}^{(k)}{\sim}x$, the slightly simpler term $\bigwedge_{k\le n} \bar{f}^{(k)}{\sim}x$ is sufficient. □

Note that the four cases are not mutually exclusive. For instance, the identity map is both normal and dually normal.

Definition 6.1.16. Let $\mathbf{A}$ be an EDHA. We will say that $\mathbf{A}$ is a *double Heyting algebra with generalised operators* if every operation in the signature of $\mathbf{A}$, not including $\{\vee, \wedge, \to, \dot{-}, 0, 1\}$, is a generalised operator.

Thus, from Theorem 6.1.15, we obtain the next result.

Corollary 6.1.17. *Let* $\mathbf{A}$ *be a double Heyting algebra with generalised operators and assume* $\mathbf{A}$ *has a finite signature. Then* $\mathbf{A}$ *has a congruence-filter term. More* *precisely, if $M = \{\dot{-}\} \cup \{f_1, \ldots, f_n\}$, then, for each $i \leq n$, let t_i be the term defined by*

$$t_i x = \begin{cases} \bigwedge_{k\leq n} f_i^{(k)} x, & \text{if } f_i \text{ is normal,} \\ \bigwedge_{k\leq n} \neg f_i^{(k)} x, & \text{else if } f_i \text{ is anti-normal,} \\ \bigwedge_{k\leq n} \neg \bar{f_i}^{(k)} {\sim} x, & \text{else if } f_i \text{ is dually normal,} \\ \bigwedge_{k\leq n} \bar{f_i}^{(k)} {\sim} x, & \text{else if } f_i \text{ is dually anti-normal.} \end{cases}$$

Then $tx = \neg{\sim}x \wedge t_1 x \wedge \ldots \wedge t_n x$ is a congruence-filter term for $\mathbf{A}$.

It it noteworthy that we have not been able to prove the existence of compatibility terms for dual normal operators and dual anti-normal operators defined on dually pseudocomplemented expansions, even in the unary case. It would seem unusual if the same compatibility terms did not work, given that they are in the appropriate language. They can be proved sufficient if the dually pseudocomplemented expansion has an underlying lattice in common with a double Heyting algebra. But for dually pseudocomplemented EHAs that do not form double Heyting algebras, we do not know of any proof.

Open Problem 8. Can Corollary 6.1.17 be generalised with $\sim$ in the signature but with $\dot{-}$ excluded?

6.2 Semisimple varieties

We now turn our attention to varieties of dually pseudocomplemented EHAs.

Definition 6.2.1. Let $\mathcal{V}$ be a variety of any signature. Then $\mathcal{V}$ is *semisimple* if every subdirectly irreducible member of $\mathcal{V}$ is simple. Let $\mathbf{A} \in \mathcal{V}$. A ternary term t in the language of $\mathcal{V}$ is called a *discriminator term on* $\mathbf{A}$ if the corresponding term function is the *discriminator function* on $\mathbf{A}$, i.e.,

$$t^{\mathbf{A}}(x, y, z) = \begin{cases} x & \text{if } x \neq y, \\ z & \text{if } x = y. \end{cases}$$

If there is a term t in the language of $\mathcal{V}$ such that t is a discriminator term on every subdirectly irreducible member of $\mathcal{V}$, we say that $\mathcal{V}$ is a *discriminator variety*.

Discriminator varieties and semisimple varieties are closely connected to varieties with EDPC. It is well known, for instance, that every discriminator variety is semisimple and has EDPC [76, 91]. A discriminator term is also a Mal'cev term, so discriminator varieties are congruence-permutable. Moreover, these three conditions characterise discriminator varieties.

Theorem 6.2.2 (Blok, Köhler, and Pigozzi [9, Corollary 3.4]). *Let $\mathcal{V}$ be a variety of any signature. The following are equivalent*:

(1) *$\mathcal{V}$ is semisimple, congruence-permutable, and has EDPC*;

(2) *$\mathcal{V}$ is a discriminator variety.*

Because EHAs have a Heyting algebra reduct, they are congruence-permutable. So Blok, Köhler, and Pigozzi's result simplifies as follows.

Corollary 6.2.3. *Let $\mathcal{V}$ be a variety of EHAs. The following are equivalent*:

(1) *$\mathcal{V}$ is semisimple and has EDPC*;

(2) *$\mathcal{V}$ is a discriminator variety.*

We will simplify this even further in the case of dually pseudocomplemented EHAs with a congruence-filter term by proving that if $\mathcal{V}$ is semisimple, then $\mathcal{V}$ has EDPC.

Definition 6.2.4. Let $\mathcal{V}$ be a variety of dually pseudocomplemented EHAs and let t be a unary term in the language of $\mathcal{V}$. We will say that t is *strongly descending* if $\mathcal{V} \models tx \leq \neg{\sim}x$.

Because $\neg{\sim}x \leq x$, a strongly descending unary term is also descending. Moreover, if t is a congruence-filter term, then the term d given by $dx = \neg{\sim}x \wedge tx$ is a strongly descending congruence-filter term, so there is no loss of generality in assuming that congruence-filter terms are strongly descending.

Definition 6.2.5. For the entirety of this section, let $\mathcal{V}$ be a fixed non-trivial variety of dually pseudocomplemented EHAs and assume that d is a strongly descending congruence-filter term on $\mathcal{V}$.

The main argument will proceed by a sequence of intermediate lemmas.

Lemma 6.2.6. *Assume that $\mathcal{V}$ is semisimple, let $\mathbf{A} \in \mathcal{V}$, let $a \in A\backslash\{1\}$, and let $\beta \in \mathrm{Con}(\mathbf{A})$. If $\mathrm{Cg}^{\mathbf{A}}(a,1)$ covers β and $d^n a > 0$, for all $n \in \omega$, then there exists $m \in \omega$ such that*:

(1) $\sim d^m a \equiv_\beta \neg d^m a$,

(2) $d^{m+1} a \equiv_\beta d^m a$,

(3) $\neg d^m a \equiv_\beta d\neg d^m a$,

(4) $\sim d^m a \equiv_\beta d{\sim}d^m a$.

Proof. Let $\alpha = \mathrm{Cg}^{\mathbf{A}}(a,1)$, assume α covers β, and assume $d^n a > 0$, for all $n \in \omega$. Let $\Gamma = \{\theta \in \mathrm{Con}(\mathbf{A}) \mid \theta \geq \beta \text{ but } \theta \not\geq \alpha\}$. Since $\mathbf{Con}(\mathbf{A})$ is complete, $\gamma := \bigvee \Gamma$ exists, and it follows from congruence-distributivity and the compactness of α that $\gamma \in \Gamma$. It is easy to see that $\mathbf{A}/\gamma$ is subdirectly irreducible, where $(\gamma \vee \alpha)/\gamma$ is the monolith, and by semisimplicity, it follows that $\mathbf{A}/\gamma$ is simple. By congruence-permutability, it follows that $\gamma \circ \alpha$ is the full congruence on $\mathbf{A}$. So $(0,1) \in \gamma \circ \alpha$, and hence there exists some $c \in A$ such that $(0,c) \in \gamma$ and $(c,1) \in \alpha$. By Theorem 5.3.2, there exists $m \in \omega$ such that $c \geq d^m a$. We then have $\neg c \leq \neg d^m a$, implying that $(\neg d^m a, 1) \in \mathrm{Cg}^{\mathbf{A}}(1, \neg c)$. Then, since $(0,c) \in \gamma$, we have that $(1, \neg c) \in \gamma$, and so $\neg d^m a \equiv_\gamma 1$. The same reasoning holds with respect to $\sim$, and so we have $\neg d^m a \equiv_\gamma {\sim} d^m a$. Moreover, since $d^m a \equiv_\alpha 1$, we have ${\sim} d^m a \equiv_\alpha 0 \equiv_\alpha \neg d^m a$. By construction, we have $\beta = \alpha \wedge \gamma$, and thus ${\sim} d^m a \equiv_\beta \neg d^m a$, proving (1).

For part (2), consider the element $d^{m+1}a \vee \neg d^m a$. We have $d^{m+1} a \equiv_\alpha 1$, and so $d^{m+1}a \vee \neg d^m a \equiv_\alpha 1$. We also have $\neg d^m a \equiv_\gamma 1$, and it follows that $d^{m+1}a \vee \neg d^m a \equiv_\gamma 1$. Then $d^{m+1}a \vee \neg d^m a \equiv_\beta 1$. Since $\sim$ is the dual pseudocomplement, we now have $d^{m+1} a \geq {\sim}\neg d^m a \pmod{\beta}$, and by part (1) we have ${\sim}\neg d^m a \equiv_\beta d^m a$. But since d is descending, we have $d^m a \geq d^{m+1} a$, and so $d^{m+1} a \equiv_\beta d^m a$.

Now for part (3), since $\neg d^m a \equiv_\gamma 1$, we have $d \neg d^m a \equiv_\gamma 1$, and since $d^m a \equiv_\alpha 1$, we have $d^m a \vee d\neg d^m a \equiv_\beta 1$. Thus, $d\neg d^m a \geq {\sim} d^m a \pmod{\beta}$. But ${\sim} d^m a \equiv_\beta \neg d^m a$ and $\neg d^m a \geq d \neg d^m a$, and therefore $d\neg d^m a \equiv_\beta \neg d^m a$. Finally, it is easy to see that (4) follows from (1) and (3). □

Lemma 6.2.7. *Assume $\mathcal{V}$ is semisimple. Then, for every $k \in \omega$, there exists $r, l \in \omega$ such that $\mathcal{V} \models d^l {\sim} d^k \neg d^r x \geq x$.*

Proof. Suppose otherwise. Then there exists a minimum $K \in \omega$ such that, for all $r, l \in \omega$, the variety $\mathcal{V}$ falsifies $d^l {\sim} d^K \neg d^r x \geq x$. Let $\mathbf{F}$ be the free algebra on one generator x in $\mathcal{V}$. For each $r \in \omega$, define the congruence

$$\theta_r = \mathrm{Cg}^{\mathbf{F}}({\sim} d^K \neg d^r x, 1).$$

Since d is descending, we have that $d^r x \geq d^{r+1} x$, for all $r \in \omega$. It follows easily that ${\sim} d^K \neg d^r x \geq {\sim} d^K \neg d^{r+1} x$. So $\{\theta_r \mid r \in \omega\}$ is an increasing chain of congruences. Now let $\alpha = \mathrm{Cg}^{\mathbf{F}}(x,1)$ and let $\Theta = \bigvee_{r \in \omega} \theta_r$.

To proceed, we will first show that the congruence Θ lies strictly between $0_{\mathbf{F}}$ and α. Suppose first that $\Theta = 0_{\mathbf{F}}$. Then, for all $r \in \omega$, we have $\theta_r = 0_{\mathbf{F}}$, which implies ${\sim} d^K \neg d^r x = 1$ in $\mathbf{F}$, and hence $\mathcal{V} \models {\sim} d^K \neg d^r x = 1$. Then in particular, $\mathcal{V} \models {\sim} d^K \neg d^r 0 = 1$, implying $\mathcal{V} \models 0 = 1$, contradicting the assumption that $\mathcal{V}$ is non-trivial. So $\Theta \neq 0_{\mathbf{F}}$. Next, by construction, we have $\alpha \geq \Theta$. Suppose, by way of contradiction, that $\Theta = \alpha$. Since α is compact and $\{\theta_r \mid r \in \omega\}$ is an increasing chain, it follows that there exists $r \in \omega$ such that $\alpha = \theta_r$. Then $x \in 1/\theta_r$, and so there exists $l \in \omega$ such that $x \geq d^l{\sim}d^K\neg d^r x$. As we are working in the free algebra, this implies that $\mathcal{V} \models x \geq d^l{\sim}d^K\neg d^r x$, contradicting the assumption that no such r and l exist, and we have proved that $0 < \Theta < \alpha$.

Now since α is compact, there exists $\beta \in \mathrm{Con}(\mathbf{F})$ such that α covers β and $\beta \geq \Theta$. Then by Lemma 6.2.6, there is some $m \in \omega$ such that $\neg d^m x \equiv_\beta d\neg d^m x$ and $\neg d^m x \equiv_\beta {\sim}d^m x$. This implies $\neg d^m x \equiv_\beta d^K\neg d^m x$ and ${\sim}\neg d^m x \equiv_\beta d^m x$. Thus we have $d^m x \equiv_\beta {\sim}\neg d^m x \equiv_\beta {\sim}d^K\neg d^m x$. But then, since $\theta_m \leq \Theta \leq \beta$, we have ${\sim}d^K\neg d^m x \equiv_\beta 1$, and so $d^m x \equiv_\beta 1$, implying $x \equiv_\beta 1$. But then $\beta \geq \alpha$, a contradiction. □

Definition 6.2.8. Assume that $\mathcal{V}$ is semisimple. Let $r\colon \omega \to \omega$ and $l\colon \omega \to \omega$ be the maps given by

$$r(i) = \min\{r \in \omega \mid (\exists l \in \omega)\ \mathcal{V} \models x \geq d^l{\sim}d^i\neg d^r x\},$$
$$l(i) = \min\{l \in \omega \mid \mathcal{V} \models x \geq d^l{\sim}d^i\neg d^{r(i)}\}.$$

By Lemma 6.2.7, both r and l are well defined.

Lemma 6.2.9. *The function r is non-decreasing.*

Proof. We will show that, for every $i \in \omega$, there exists some $l \in \omega$ such that $\mathcal{V} \models x \geq d^l{\sim}d^i\neg d^{r(i+1)}$, and it will then follow by definition that $r(i+1) \geq r(i)$. Since d is descending, $d^{i+1}\neg d^{r(i+1)}x \leq d^i\neg d^{r(i+1)}x$, so ${\sim}d^{i+1}\neg d^{r(i+1)}x \geq {\sim}d^i\neg d^{r(i+1)}x$. By the definition of r, there exists $l \in \omega$ such that $x \geq d^l{\sim}d^{i+1}\neg d^{r(i+1)}$, implying $x \geq d^l{\sim}d^i\neg d^{r(i+1)}x$, as required. □

Lemma 6.2.10. *Assume that $\mathcal{V}$ is semisimple and $\mathcal{V}$ falsifies $d^{n+1}x = d^n x$, for all $n \in \omega$. Then, for each $i \in \omega$, there is a simple algebra $\mathbf{A}_i \in \mathcal{V}$ and an element $a_i \in A_i$ such that $d^{r(i)}a_i > 0$ and $d^{r(i)+1}a_i = 0$.*

Proof. Firstly, for all simple $\mathbf{A} \in \mathcal{V}$ and all $a \in A$, by Theorem 5.3.3, there exists $m \in \omega$ such that $d^m a = 0$. In what follows, let m_a denote the smallest such m. We will leave the dependence of m_a on the algebra $\mathbf{A}$ implicit.

Now suppose the lemma does not hold. Then there exists $i \in \omega$ such that, for every simple algebra $\mathbf{A} \in \mathcal{V}$ and all $a \in A$, if $d^{r(i)+1}a = 0$, then $d^{r(i)}a = 0$. If, for all simple $\mathbf{A} \in \mathcal{V}$ and all $a \in A$, we have $m_a \leq r(i)$, then $d^{r(i)}a = 0$, for all $a \in A$. But then $\mathcal{V} \models d^{r(i)+1}x = d^{r(i)}x$, a contradiction. Thus there exists a simple algebra $\mathbf{A} \in \mathcal{V}$ and some $a \in A$ such that $m_a > r(i)$. Then there exists $k \in \omega$ such that

$m_a = k + r(i) + 1$, and hence $d^{m_a}a = d^{r(i)+1}d^k a = 0$. By assumption, it follows that $d^{r(i)}d^k a = 0$, that is, $d^{r(i)+k}a = 0$. But $r(i) + k < m_a$, contradicting the minimality of m_a. □

Lemma 6.2.11. *Assume that $\mathcal{V}$ is semisimple and $\mathcal{V}$ falsifies $d^{n+1}x = d^n x$, for all $n \in \omega$. Then, for each $i \in \omega$, there is a simple algebra $\mathbf{A}_i \in \mathcal{V}$ and an element $b_i \in A_i$ such that, for all $k \in \omega$, if $i \geq k$, then $d^k b_i > 0$ and $d^{l(k)+r(k)+1}{\sim}d^k b_i = 0$.*

Proof. By Lemma 6.2.10, for each $i \in \omega$, there exists a simple algebra $\mathbf{A}_i \in \mathcal{V}$ and an element $a_i \in A_i$ such that $d^{r(i)}a_i > 0$ and $d^{r(i)+1}a_i = 0$. Now let $b_i = \neg d^{r(i)}a_i$, let $k \in \omega$, and let $i \geq k$. If $d^k b_i = 0$, then, since $i \geq k$, we have $d^i b_i = 0$. That is, $d^i \neg d^{r(i)}a_i = 0$. Then ${\sim}d^i \neg d^{r(i)}a_i = 1$, and repeated applications of d to both sides gives $d^{l(i)}{\sim}d^i\neg d^{r(i)}a_i = 1$. By definition of l and r, we have $a_i \geq d^{l(i)}{\sim}d^i\neg d^{r(i)}a_i$, so $a_i = 1$. But then $d^{r(i)+1}a_i = 1$, contradicting the choice of a_i. Thus, we must have $d^k b_i > 0$.

To show that $d^{l(k)+r(k)+1}{\sim}d^k b_i = 0$ holds, first observe that since r is non-decreasing, we have

$$\begin{aligned} d^{l(k)+r(k)+1}{\sim}d^k b_i &= d^{l(k)+r(k)+1}{\sim}d^k\neg d^{r(i)}a_i \\ &= d^{l(k)+r(k)+1}{\sim}d^k\neg d^{r(k)}d^{r(i)-r(k)}a_i. \end{aligned}$$

From the inequality $x \geq d^{l(k)}{\sim}d^k\neg d^{r(k)}x$, applying d repeatedly on both sides gives $d^{r(k)+1}x \geq d^{l(k)+r(k)+1}{\sim}d^k\neg d^{r(k)}x$. Substituting $d^{r(i)-r(k)}a_i$ for x then gives

$$d^{r(i)+1}a_i = d^{r(k)+1}d^{r(i)-r(k)}a_i \geq d^{l(k)+r(k)+1}{\sim}d^k\neg d^{r(k)}d^{r(i)-r(k)}a_i,$$

and since $d^{r(i)+1}a_i = 0$, we have that $d^{l(k)+r(k)+1}{\sim}d^k b_i = 0$, as required. □

Theorem 6.2.12. *If $\mathcal{V}$ is semisimple, then $\mathcal{V} \models d^{n+1}x = d^n x$, for some $n \in \omega$.*

Proof. Suppose that $\mathcal{V}$ is semisimple and falsifies $d^{n+1}x = d^n x$, for all $n \in \omega$. By Lemma 6.2.11, for every $i \in \omega$, there exists a simple algebra $\mathbf{A}_i \in \mathcal{V}$ and an element $b_i \in A_i$ such that, for all $k \in \omega$, if $i \geq k$, then $d^k b_i > 0$ and $d^{l(k)+r(k)+1}{\sim}d^k b_i = 0$.

Let U be a non-principal ultrafilter on ω, let $\mathbf{A}$ be the ultraproduct $\prod_{i\in\omega}\mathbf{A}_i/U$, and let $b = \langle b_i \mid i \in \omega\rangle/U$. By the properties of ultraproducts, we then have that $d^k b > 0$ and $d^{l(k)+r(k)+1}{\sim}d^k b = 0$, for all $k \in \omega$. Let $\alpha = \mathrm{Cg}^{\mathbf{A}}(b, 1)$ and let β be a lower cover of α. By Lemma 6.2.6, there exists $m \in \omega$ such that ${\sim}d^m b \equiv_\beta d{\sim}d^m b$, and so ${\sim}d^m b \equiv_b d^{l(m)+r(m)+1}{\sim}d^m b = 0$. Hence $d^m b \equiv_\beta 1$, and so $b \equiv_\beta 1$, implying $\beta \geq \alpha$, a contradiction. □

Using Theorem 5.4.2 we then get the desired result.

Corollary 6.2.13. *If $\mathcal{V}$ is semisimple, then $\mathcal{V}$ has EDPC.*

Corollary 6.2.14. *Let $\mathcal{V}$ be a variety of dually pseudocomplemented EHAs with a congruence-filter term. The following are equivalent:*

(1) *$\mathcal{V}$ is semisimple;*

(2) *$\mathcal{V}$ is a discriminator variety.*

Note that the discriminator term is not given explicitly.

6.3 Discriminator varieties

In this section we will extend the previous result by giving an equational characterisation of discriminator varieties. The discriminator term is given explicitly.

Theorem 6.3.1. *Let $\mathcal{V}$ be a variety of dually pseudocomplemented EHAs and assume $\mathcal{V}$ has a strongly descending congruence-filter term d. The following are equivalent:*

(1) *$\mathcal{V}$ is semisimple;*

(2) *$\mathcal{V}$ is a discriminator variety;*

(3) *$\mathcal{V}$ is a discriminator variety with discriminator term*

$$[z \wedge d^n(x \leftrightarrow y)] \vee [x \wedge {\sim} d^n(x \leftrightarrow y)],$$

for some $n \in \omega$;

(4) *$\mathcal{V}$ has DPC and $\mathcal{V} \models x \leq d{\sim}d^n\neg x$, for some $n \in \omega$;*

(5) *$\mathcal{V}$ has EDPC and $\mathcal{V} \models x \leq d{\sim}d^n\neg x$, for some $n \in \omega$;*

(6) *$\mathcal{V} \models d^{n+1}x = d^n x$ and $\mathcal{V} \models x \leq d{\sim}d^n\neg x$, for some $n \in \omega$;*

(7) *$\mathcal{V} \models d^{n+1}x = d^n x$ and $\mathcal{V} \models d{\sim}d^n x = {\sim}d^n x$, for some $n \in \omega$.*

Proof. We will show that (2) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (7) $\Rightarrow$ (3) $\Rightarrow$ (2), and since we just proved the equivalence of (1) and (2), and Theorem 5.4.2 shows that (4) and (5) are equivalent, the result will hold.

It is obvious that (3) implies (2), and it follows from Theorem 5.4.2 that (6) implies (5). To show that (2) implies (6), assume that $\mathcal{V}$ is a discriminator variety. By Theorem 6.2.2, it follows that $\mathcal{V}$ is semisimple and has EDPC, and then Theorem 5.4.2 implies there exists $n \in \omega$ such that $\mathcal{V} \models d^{n+1}x = d^n x$. We now show that $\mathcal{V} \models d{\sim}d^n\neg x \geq x$. Let $\mathbf{A} \in \mathcal{V}$ be subdirectly irreducible. Since $\mathcal{V}$ is a semisimple, $\mathbf{A}$ is simple. As $\mathcal{V} \models d^{n+1}x = d^n x$, we also have $\mathcal{V} \models d^{n+1}\neg x = d^n\neg x$, and so $\mathrm{Fg}^{\mathbf{A}}(\neg x) = {\uparrow}d^n\neg x$, for all $x \in A$. Let $x \in A$. Since $\mathbf{A}$ is simple, we have that $d^n\neg x \in \{0, 1\}$. It is easily verified that $d{\sim}d^n\neg 0 = 0$, so the inequality holds when $x = 0$. Now assume $x > 0$. Since $d^n\neg x = 1$ if and only if $\neg x = 1$, and $\neg x = 1$ if and only if $x = 0$, it follows that $d^n\neg x = 0$, and thus $d{\sim}d^n\neg x = 1 \geq x$. As $\mathbf{A}$ was an arbitrary subdirectly irreducible algebra in $\mathcal{V}$, it follows that $\mathcal{V} \models x \leq d{\sim}d^n\neg x$.

We now prove that (5) implies (7). Assume that (5) holds. First, by Theorem 5.4.2, there exists $n \in \omega$ such that $\mathcal{V} \models d^{n+1}x = d^n x$. Let $x \in A$. Clearly $d{\sim}d^n 1 = {\sim}d^n 1$, so assume $x \neq 1$. Since d is strongly descending, it follows that $\neg{\sim}d^n x \geq d(d^n x) = d^n x$, and since $d^n x \geq \neg{\sim}d^n x$, we have $\neg{\sim}d^n x = d^n x$. Then by assumption, there exists $m \in \omega$ such that ${\sim}d^n x \leq d{\sim}d^m\neg{\sim}d^n x$. Since $\neg{\sim}d^n x = d^n x$, we have ${\sim}d^n x \leq d{\sim}d^m d^n x = d{\sim}d^n x$, and since d is descending, we have $d{\sim}d^n x \leq {\sim}d^n x$. Hence ${\sim}d^n x = d{\sim}d^n x$.

Finally, we prove that (7) implies (3). Assume that (7) holds. Let $\mathbf{A} \in \mathcal{V}$, assume $\mathbf{A}$ is subdirectly irreducible, and let $x \in A$. By assumption, there exists $n \in \omega$ such that $d^{n+1}x = d^n x$ and $d{\sim}d^n x = {\sim}d^n x$. Hence $\mathrm{Fg}^{\mathbf{A}}(d^n x) = {\uparrow}d^n x$ and $\mathrm{Fg}^{\mathbf{A}}({\sim}d^n x) = {\uparrow}{\sim}d^n x$. By definition of $\sim$, we have $d^n x \vee {\sim}d^n x = 1$, and so $\mathrm{Fg}^{\mathbf{A}}(d^n x) \cap \mathrm{Fg}^{\mathbf{A}}({\sim}d^n x) = \{1\}$. If $d^n x \notin \{0, 1\}$, this contradicts subdirect irreducibility, so we must have $d^n x \in \{0, 1\}$. Hence,

$$d^n x = \begin{cases} 1, & \text{if } x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

It is then easily verified that the term given by

$$[z \wedge d^n(x \leftrightarrow y)] \vee [x \wedge {\sim}d^n(x \leftrightarrow y)]$$

is a discriminator term on $\mathcal{V}$. □

It is interesting that results similar to Theorem 6.3.1 exist for certain residuated lattices. For example, Kowalski [60] proved that all semisimple varieties of $\mathrm{FL}_{\mathrm{ew}}$ algebras are discriminator varieties, which was extended by Takamura [83] to bounded weak-commutative residuated lattices with an S4-like modal operator. In an unpublished manuscript, Kowalski and Ferreirim [61] proved that the class of Hamiltonian residuated lattices also has the property that semisimple varieties are discriminator varieties. For more on Hamiltonian residuated lattices, see [16]. For each of those results, it is proved that a subvariety of the corresponding class has EDPC if and only if it satisfies the equation $(x \wedge 1)^n = (x \wedge 1)^{n+1}$, for some $n \in \omega$, and that semisimple subvarieties satisfy the equation. This is of remarkable similarity to the current setting (cf. Theorem 5.4.2 and Theorem 6.2.12). We believe that this is no coincidence, and the hunt for a grand unifying theory continues.

Open Problem 9. Generalise Theorem 6.3.1 so that it incorporates the results for the residuated lattices considered by Kowalski [60], Takamura [83], and Kowalski and Ferreirim [61].

7

Examples of congruence-filter terms

The most basic example of a congruence-filter term is the identity term on Heyting algebras, so the results of Chapter 5 apply to the variety of Heyting algebras, albeit rather trivially. This can be considered more generally. For an algebra $\mathbf{A}$, a *compatible operation* on $\mathbf{A}$ is a function $f\colon A^n \to A$ that preserves the congruence relations of $\mathbf{A}$. Definability conditions for compatible operations on Heyting algebras have been studied by Caicedo and Cignoli [19] and Biraben and Martín [30]. This includes Heyting algebras with successor, which were introduced by Kuznetsov [65] as an equational generalisation of finite Heyting chains with a unary successor operation (see also [21, 22]). Trivially, the identity function on Heyting algebras equipped with compatible operations is a congruence-filter term.

Our treatment is in the opposite direction, where there are fewer congruences on the algebra when compared to its Heyting algebra reduct. In this chapter, we collate some examples from the literature of EHAs with a non-trivial congruence-filter term. In the published results we often find a proof of the following generic statement: "$\theta(F)$ is a congruence if and only if F is closed under t", where t is some term in the language of the algebra. The results in this thesis collapse these proofs into a single general framework, occasionally requiring only a minor computation to find the term t. As a consequence, some known results are given as corollaries. We also obtain some new results for De Morgan–Heyting algebras and symmetric Heyting relation algebras.

7.1 Double Heyting algebras

As we have seen from the direct proofs by Sankappanavar and Köhler, as well as from Corollary 5.2.15 and Lemma 5.2.16, the term d given by $dx = \neg{\sim}x$ is a congruence-filter term on $\mathcal{H}^+$ and $\mathcal{DH}$. The statement of Theorem 6.3.1 simplifies for these varieties. The next result was first proved directly by the author in [85].

Theorem 7.1.1 (Taylor [85]). *Let $\mathcal{V}$ be a variety of regular double p-algebras, H^+-algebras, or double Heyting algebras. The following are equivalent*:

(1) *$\mathcal{V}$ is semisimple*;

(2) *$\mathcal{V}$ is a discriminator variety*;

(3) *$\mathcal{V}$ has DPC*;

(4) *$\mathcal{V}$ has EDPC*;

(5) *$\mathcal{V} \models d^{n+1}x = d^n x$, for some $n \in \omega$.*

Proof. We show that $\mathcal{H}^+ \models x \leq d{\sim}d^n\neg x$, for all $n \in \omega$. The same argument applies for $\mathcal{DH}$ and $\mathcal{RDP}$. Since ${\sim}{\sim}\neg x \leq \neg x$, we have $d{\sim}\neg x = \neg{\sim}{\sim}\neg x \geq \neg\neg x \geq x$. Then $d^n\neg x \leq \neg x$ implies ${\sim}d^n\neg x \geq {\sim}\neg x$, so $d{\sim}d^n\neg x \geq d{\sim}\neg x \geq x$, as required. □

Köhler [59] gave an example of a subdirectly irreducible double Heyting algebra that is not simple, so $\mathcal{DH}$ is not semisimple, and it follows that $\mathcal{DH}$ is not a discriminator variety, nor does it have DPC or EDPC. This can be proved in a different way by considering fences. It is easy to see that if F is a fence with $2(n+1)$ elements, then $\mathcal{U}(F)$ fails to satisfy the equation $d^{n+1}x = d^n x$. The same thing applies to H$^+$-algebras and regular double p-algebras. This also provides a non-constructive proof that there exists a subdirectly irreducible double Heyting algebra that is not simple.

Another subvariety of double Heyting algebras we may consider is the variety of *Stone double Heyting algebras*, i.e., double Heyting algebras satisfying the equation $\neg x \vee \neg\neg x = 1$. Iturrioz [49] proved that the variety of Stone double Heyting algebras is semisimple. We now prove it as a corollary.

Proposition 7.1.2. *Let* **A** *be a Stone double Heyting algebra. Then $d^2x = dx$, for all $x \in A$.*

Proof. Let $x \in A$. Since $\neg x \vee \neg\neg x = 1$, we have $\neg\neg x \geq {\sim}\neg x$ by definition of $\sim$. Then ${\sim}\neg x \geq \neg\neg x$ implies $\neg\neg x = {\sim}\neg x$, so $d^2x = \neg{\sim}\neg{\sim}x = \neg\neg\neg{\sim}x = \neg{\sim}x$, as claimed. □

Corollary 7.1.3 (Iturrioz [49]). *The variety of Stone double Heyting algebras is semisimple.*

7.2 Heyting algebras with operators

We gave a cursory description of Boolean algebras with operators in Chapter 5. They were introduced by Jónsson and Tarski [54] as a generalisation of relation algebras, and surveys can be found in Goldblatt [38] or Jónsson [53]. We have mentioned that they are traditionally defined with join-preserving operations, but in this thesis we have defined them as meet-preserving, to allow a smooth transition from a Boolean reduct to a Heyting algebra reduct. The symmetry of Boolean algebras means the distinction is irrelevant. But since Boolean algebras also have a double Heyting algebra term reduct, Corollary 6.1.17 applies. Hence, Boolean algebras with a mixed signature of join-preserving, join-reversing, meet-preserving, and meet-reversing operations have a congruence-filter term. For that reason, let us reconsider BAOs more generally.

Definition 7.2.1. An algebra $\mathbf{A} = \langle A; \{f_i \mid i \in I\}, \vee, \wedge, \neg, 0, 1\rangle$ is a *Boolean algebra with operators* (BAO for short) if $\langle A; \vee, \wedge, \neg, 0, 1\rangle$ is a Boolean algebra and each f_i is a normal operator, an anti-normal operator, a dual normal operator, or a dual anti-normal operator (refer to Definitions 5.2.1 and 6.1.14). If each f_i is a unary normal operator, then we will say that $\mathbf{A}$ is a *unary normal BAO*. A variety $\mathcal{V}$ of unary normal BAOs is called *cylic* if, for every operator f in the signature of $\mathcal{V}$, there is a unary term t in the language of $\mathcal{V}$ such that $\mathcal{V} \models x \leq f\neg t\neg x$.

Let $\mathcal{V}$ be a variety of unary normal BAOs and assume $\mathcal{V}$ has a finite signature. Let $f_1, \dots, f_k$ be the operators. The term $dx = x \wedge f_1 x \wedge \dots \wedge f_k x$ is a congruence-filter term for $\mathcal{V}$ by Corollary 5.2.15. One can prove that $\mathcal{V}$ is cyclic if and only if $\mathcal{V} \models x \leq d\neg d^n \neg x$, for some $n \in \omega$ (see [62, Proposition 6]). We generalise this in the next definition.

Definition 7.2.2. Let $\mathcal{V}$ be a variety of BAOs. We say that $\mathcal{V}$ is *cyclic* if there exists a congruence-filter term d on $\mathcal{V}$ such that $\mathcal{V} \models x \leq d\neg d^n \neg x$, for some $n \in \omega$.

Recall that our proof of Theorem 6.3.1 is based on the proof of Kowalski and Kracht [62] for unary normal BAOs. It is easily seen that Kowalski and Kracht's proof applies verbatim to BAOs with a congruence-filter term. We obtain it as a corollary of Theorem 6.3.1.

Theorem 7.2.3 (Kowalski and Kracht [62])**.** *Let $\mathcal{V}$ be a variety of BAOs with a congruence-filter term. The following are equivalent:*

(1) *$\mathcal{V}$ is semisimple;*

(2) *$\mathcal{V}$ is a discriminator variety;*

(3) *$\mathcal{V}$ is cyclic and has DPC;*

(4) *$\mathcal{V}$ is cyclic and has EDPC.*

In Definition 5.2.14, we say that a Heyting algebra with operators (HAO) is an EHA equipped with normal operators and anti-normal operators. Notice that this excludes join-preserving and join-reversing operations, so HAOs do not generalise BAOs. However, there are EHAs that could be fairly described as generalised BAOs, but are not considered HAOs. For example, monadic Heyting algebras, called bi-topological pseudo-Boolean algebras by Ono [72], include a meet-preserving operation as well as a join-preserving one. They are not guaranteed a congruence-filter term by our general methods so far, but they still possess one.

Definition 7.2.4. An algebra $\mathbf{A} = \langle A; \vee, \wedge, \to, \forall, \exists, 0, 1\rangle$ is a *monadic Heyting algebra* if the reduct $\langle A; \vee, \wedge, \to, 0, 1\rangle$ is a Heyting algebra and $\forall$ and $\exists$ are unary operations satisfying, for all $x, y \in A$,

(1) $\forall x \leq x$,
(2) $x \leq \exists x$,
(3) $\forall(x \wedge y) = \forall x \wedge \forall y$,
(4) $\exists(x \vee y) = \exists x \vee \exists y$,
(5) $\forall 1 = 1$,
(6) $\exists 0 = 0$,
(7) $\forall\exists x = \exists x$,
(8) $\exists\forall x = \forall x$,
(9) $\exists(\exists x \wedge y) = \exists x \wedge \exists y$.

We will denote the variety of monadic Heyting algebras by $\mathcal{MH}$.

Monadic Heyting algebras were studied extensively by Bezhanishvili [5–8]. We do not generalise any of Bezhanishvili's results except for the characterisation of congruences. The next lemma extracts the essential information.

Lemma 7.2.5. *Let* $\mathbf{A}$ *be an* EHA*, let* f *be an operation on* $\mathbf{A}$*, and let* g *be a unary operation on* $\mathbf{A}$*. Assume that* t *is a compatibility term for* f *and that* $t(x \to y) \leq gx \to gy$*, for all* $x, y \in A$*. Then* t *is a compatibility term for* $\{f, g\}$*.*

Proof. It suffices to show that if F is a filter of $\mathbf{A}$, then F is compatible with f if and only if F is compatible with $\{f, g\}$. One direction is trivial. For the non-trivial direction, assume F is compatible with f and let $x \leftrightarrow y \in F$. By assumption, F is closed under t, so $t(x \leftrightarrow y) \in F$. We then have

$$t(x \leftrightarrow y) \leq t(x \to y) \wedge t(y \to x) \leq (gx \to gy) \wedge (gy \to gx) = gx \leftrightarrow gy.$$

Hence F is also compatible with g. □

Proposition 7.2.6 (Bezhanishvili [5])**.** *Let* $\mathbf{A}$ *be a monadic Heyting algebra and let* F *be a filter of* $\mathbf{A}$*. Then* $\theta(F)$ *is a congruence on* $\mathbf{A}$ *if and only if* F *is closed under* $\forall$*.*

Proof. Since $\forall$ is a normal operator, we have $[\forall] = \forall$. In [5] it is stated without proof that, combined with the remaining axioms above, the equation $\exists(\exists x \wedge y) = \exists x \wedge \exists y$ is equivalent to $\forall(x \to y) \leq \exists x \to \exists y$. The proof of that fact is not difficult, and the result then holds by Lemma 7.2.5. □

It is easy to show that $\mathcal{MH} \models \forall\forall x = \forall x$. Applying Theorem 5.4.2, we then see that the variety of monadic Heyting algebras has EDPC, where the equation given by $a \leftrightarrow b \geq \forall(c \leftrightarrow d)$ defines principal congruences. But they are not dually pseudocomplemented, so we cannot apply Theorem 6.3.1. Nonetheless, Bezhanishvili has characterised semisimple monadic Heyting algebras.

Theorem 7.2.7 (Bezhanishvili [5])**.** *A variety $\mathcal{V}$ of monadic Heyting algebras is semisimple if and only if $\mathcal{V} \models \exists x = \neg\forall\neg x$.*

It follows from Corollary 6.2.3 and the fact that monadic Heyting algebras have EDPC that semisimple varieties of monadic Heyting algebras are discriminator varieties. This was not noted by Bezhanishvili in [5–8]. In fact, it is easy to see from the fact that $\forall$ is idempotent that the term

$$[z \wedge \forall(x \leftrightarrow y)] \vee [x \wedge \forall\neg(x \leftrightarrow y)]$$

is a discriminator term for semisimple varieties of monadic Heyting algebras. Hence, a variety of monadic Heyting algebras is semisimple if and only if it is a discriminator variety.

Open Problem 10. Every semisimple variety of monadic Heyting algebras is a discriminator variety. But they are not dually pseudocomplemented, so this fact does not follow from Theorem 6.3.1. With that in mind, find a common generalisation, perhaps also including the residuated lattices mentioned in Open Problem 9.

7.3 De Morgan–Heyting algebras

Definition 7.3.1. An algebra $\mathbf{A} = \langle A; \vee, \wedge, \to, \frown, 0, 1\rangle$ is an *Ockham–Heyting algebra* if the reduct $\langle A; \vee, \wedge, \to, 0, 1\rangle$ is a Heyting algebra and $\frown$ is a dual bounded lattice endomorphism, i.e., for all $x, y \in A$,

(1) $\frown(x \wedge y) = \frown x \vee \frown y$,

(2) $\frown(x \vee y) = \frown x \wedge \frown y$,

(3) $\frown 1 = 0$,

(4) $\frown 0 = 1$.

An algebra $\mathbf{A}$ is a *De Morgan–Heyting algebra* if it is an Ockham–Heyting algebra such that $\frown\frown x = x$, for all $x \in A$. In other words, a De Morgan–Heyting algebra is an Ockham–Heyting algebra such that $\frown$ is an involutive dual lattice automorphism. Let $\mathcal{OH}$ denote the variety of Ockham–Heyting algebras and let $\mathcal{DMH}$ denote the variety of De Morgan–Heyting algebras.

Ockham–Heyting algebras were introduced by Sankappanavar in [80], where it was also proved that their congruences are determined by filters closed under $\neg\frown$. De Morgan–Heyting algebras were introduced by Monteiro [71], and they have appeared in the literature under other names, such as symmetric Heyting algebras [34, 81] and Heyting algebras with involution [70]. Meskhi [70] proved directly that congruences on a De Morgan–Heyting algebra are given by filters closed under $\neg\frown$, which also follows from Sankappanavar's result. Clearly, $\frown$ is an anti-normal anti-operator, so Ockham–Heyting algebras are HAOs, and the characterisation of congruence-filters follows from Corollary 5.2.15.

Proposition 7.3.2 (Sankappanavar [80])**.** *Let* $\mathbf{A}$ *be an Ockham–Heyting algebra and let* F *be a filter of* $\mathbf{A}$*. Then* $\theta(F)$ *is a congruence on* $\mathbf{A}$ *if and only if* F *is closed under* $\neg\frown$*.*

Meskhi investigated the subvariety of $\mathcal{DMH}$ satisfying the identity $\neg\neg x = \frown\neg x$. Denote this subvariety by $\mathcal{SHR}$, short for "symmetric Heyting algebra with a regular involution" as described in [70]. It is shown in [70] that $\mathcal{SHR}$ is a discriminator variety. Meskhi's result was given in more generality by Sankappanavar [80], who proved that subvarieties of $\mathcal{DMH}$ satisfying $t^{n+1}x = t^n x$, for some $n \in \omega$, are discriminator varieties, where the term t is defined by $tx = x \wedge \neg\frown x$. These subvarieties were studied further by Castaño and Santis [20]. The next proposition shows that Meskhi's result is a special case of Sankappanavar's result.

Proposition 7.3.3. *The variety* $\mathcal{SHR}$ *satisfies the equation* $t^2 x = tx$*.*

Proof. Let $\mathbf{A} \in \mathcal{SHR}$ and let $x \in A$. We have $\neg\frown\neg\frown x = \neg\neg\neg\frown x = \neg\frown x$, so $t^2 x = \neg\frown\neg\frown x \wedge \neg\frown x \wedge x = \neg\frown x \wedge x = tx$, as claimed. □

A valuable observation at this point is that De Morgan–Heyting algebras have a double Heyting algebra term reduct via the term $y \dot{-} x = \frown(\frown x \to \frown y)$. This is not true for Ockham–Heyting algebras in general. For De Morgan–Heyting algebras, we then have a dual pseudocomplement term-definable by ${\sim}x = \frown\neg\frown x$. Now let d be the $\mathcal{DMH}$-term given by $dx = \neg{\sim}x \wedge \neg\frown x$. Note that d is a strongly descending congruence-filter term. Thus, Theorem 6.3.1 is applicable. Just as we did for double

Heyting algebras, we can simplify the statement. The next result does not seem to appear in the literature.

Theorem 7.3.4. *Let $\mathcal{V}$ be a subvariety of $\mathcal{DMH}$. The following are equivalent:*

(1) *$\mathcal{V}$ is semisimple;*

(2) *$\mathcal{V}$ is a discriminator variety;*

(3) *$\mathcal{V}$ has DPC;*

(4) *$\mathcal{V}$ has EDPC;*

(5) *$\mathcal{V} \models d^{n+1}x = d^n x$, for some $n \in \omega$.*

Proof. According to Theorem 6.3.1, it is sufficient to prove that, for all $n \in \omega$,

$$\mathcal{V} \models d^{n+1}x = d^n x \implies \mathcal{V} \models d{\sim}d^n x = {\sim}d^n x.$$

Let $\mathbf{A} \in \mathcal{V}$ and let $x \in A$. Assume that $d^{n+1}x = d^n x$ and let $a = d^n x$. Then $da = a$. By definition of d, this means that $\neg{\sim}a \wedge \neg{\frown}a = a$. Then $a \leq \neg{\frown}a$, which implies by the definition of $\neg$ that $a \wedge {\frown}a = 0$. So ${\frown}a \vee a = 1$, and it follows that a and ${\frown}a$ are mutual complements. We then have ${\sim}d^n x = {\sim}a = {\frown}a = \neg a$, and hence

$$d{\sim}d^n x = d{\sim}a = d{\frown}a = \neg{\sim}{\frown}a \wedge \neg{\frown}{\frown}a = {\frown}a \wedge {\frown}a = {\frown}a,$$

which completes the proof. □

It is known that the variety $\mathcal{DMH}$ is not itself semisimple—in [34], a subdirectly irreducible De Morgan–Heyting algebra that is not simple is constructed.

7.4 Symmetric Heyting relation algebras

Definition 7.4.1. An algebra $\mathbf{A} = \langle A; \vee, \wedge, \rightarrow, \circ, {\frown}, 0, 1, \mathrm{id}\rangle$ is a *symmetric Heyting relation algebra* (SHRA for short) if

(1) $\langle A; \vee, \wedge, \rightarrow, {\frown}, 0, 1\rangle$ is a De Morgan–Heyting algebra,

(2) $\langle A; \circ, \mathrm{id}\rangle$ is a monoid,

(3) for all $x, y, z \in A$,

  (i) ${\sim}{\frown}(x \circ y) \leq ({\sim}{\frown}y) \circ ({\sim}{\frown}x)$,

  (ii) $x \circ y \leq z \iff x \leq {\frown}(y \circ {\frown}z)$.

Note that the third axiom depends on the double Heyting algebra term reduct. Define the *right converse* by ${\smile}x = \neg{\frown}x$ and the *left converse* by ${\smile}x = {\sim}{\frown}x$. Let $\mathcal{SHRA}$ denote the class of SHRAs.

It is worth noting that the two converses form a Galois connection. Notice too that (3ii) says $x/y = \frown(x \circ \frown y)$ is the left residual of $\circ$. Since residuated lattices form an equational class, it follows that (3ii) is equivalent to a set of equations, and hence $\mathcal{SHRA}$ is a variety. There is also a right residual of $\circ$, given by $x \backslash y = \frown(\frown y \circ x)$. The class of SHRAs is therefore interesting from the perspectives of both residuated lattices and expansions of double Heyting algebras. Symmetric Heyting relation algebras extract basic algebraic properties of the collection of binary relations on an incidence structure, specifically those relations that satisfy a certain condition consistent with the incidence relation (see [81] for more detail). This includes binary relations on sets as a special case. In that sense, they are a natural generalisation of Tarski's relation algebras.

Definition 7.4.2. A *relation algebra* is an algebra $\mathbf{B} = \langle B; \vee, \wedge, \circ, \neg, \smile, 0, 1, \mathrm{id} \rangle$ such that

(1) $\langle B; \vee, \wedge, \neg, 0, 1 \rangle$ is a Boolean algebra,

(2) $\langle B; \circ, \mathrm{id} \rangle$ is a monoid,

(3) for all $x, y, z \in B$, we have $x \circ y \leq \neg z \Leftrightarrow \smile x \circ z \leq \neg y \Leftrightarrow z \circ \smile y \leq \neg x$.

The operation $\smile$ is called the *converse*.

The third condition is equivalent to an equation (see [47]), so the class of relation algebras is a variety. The underlying lattice of an SHRA is Boolean if and only if it satisfies $\curvearrowleft x = \curvearrowright x$, in which case the traditional relation algebra converse operation is given by $\smile x = \neg \frown x$. Moreover, a relation algebra defines an SHRA via $\frown x = \smile \neg x$. Hence, the class of relation algebras is term-equivalent to a subvariety of symmetric Heyting relation algebras. For more detail, see Stell [81].

SHRAs have not been extensively studied, apparently only by Stell [81], who gave them their name. In a personal communication, Peter Jipsen raised the question of whether $\mathcal{SHRA}$ is a discriminator variety. Our results apply in this case. It is not hard to prove that $\circ$ is a dual operator, and because of the double Heyting algebra term reduct, Corollary 6.1.17 applies. Hence, the term d given by

$$dx = \neg{\sim}x \wedge \neg\frown x \wedge \neg(1 \circ {\sim}x) \wedge \neg({\sim}x \circ 1)$$

is a congruence-filter term for $\mathcal{SHRA}$. Stell does not consider congruences on SHRAs, so the following results are new.

Proposition 7.4.3. *Let* $\mathbf{A}$ *be a symmetric Heyting relation algebra, let* F *be a filter of* $\mathbf{A}$*, and let* d *be the term defined by* $dx = \neg{\sim}x \wedge \neg\frown x \wedge \neg(1 \circ {\sim}x) \wedge \neg({\sim}x \circ 1)$*. Then* $\theta(F)$ *is a congruence of* $\mathbf{A}$ *if and only if* F *is closed under* d*.*

Theorem 7.4.4. *Let $\mathcal{V}$ be a variety of symmetric Heyting relation algebras and let d be the term defined by $dx = \neg{\sim}x \wedge \neg{\frown}x \wedge \neg(1 \circ {\sim}x) \wedge \neg({\sim}x \circ 1)$. The following are equivalent:*

(1) *$\mathcal{V}$ is semisimple;*

(2) *$\mathcal{V}$ is a discriminator variety;*

(3) *$\mathcal{V}$ has DPC and $\mathcal{V} \models x \leq d{\sim}d^n\neg x$, for some $n \in \omega$;*

(4) *$\mathcal{V}$ has EDPC and $\mathcal{V} \models x \leq d{\sim}d^n\neg x$, for some $n \in \omega$;*

(5) *$\mathcal{V} \models d^{n+1}x = d^n x$ and $\mathcal{V} \models x \leq d{\sim}d^n\neg x$, for some $n \in \omega$;*

(6) *$\mathcal{V} \models d^{n+1}x = d^n x$ and $\mathcal{V} \models d{\sim}d^n x = {\sim}d^n x$, for some $n \in \omega$.*

Unlike the earlier results, we have not been able to simplify the characterisation for SHRAs. The question of whether $\mathcal{SHRA}$ is a discriminator variety is raised in comparison to the variety of relation algebras, which is a known discriminator variety (see [47, Theorem 3.17]). Although we have the characterisation above, at the point of writing we do not know whether or not $\mathcal{SHRA}$ itself is a discriminator variety.

Open Problem 11. Is $\mathcal{SHRA}$ a discriminator variety? One way to prove otherwise is to observe that, because of the double Heyting algebra reduct, finite subdirectly irreducible SHRAs are simple. Then, to show that $\mathcal{SHRA}$ is not a discriminator variety, it would be sufficient to exhibit, for each $n \in \omega$, a finite SHRA—perhaps based on a fence—that does not satisfy $d^n x = 0$.

The author suspected that these examples could be found by assuming that $\circ = \wedge$, which would simplify the congruence-filter term. It turns out that this is not a helpful assumption.

Proposition 7.4.5. *Let $\mathbf{A}$ be a SHRA and assume $\mathbf{A} \models x \circ y = x \wedge y$. Then $\mathbf{A}$ is Boolean and $\mathbf{A} \models {\frown}x = \neg x$.*

Proof. Because ${\frown}(x \circ {\frown}y)$ is the left residual of $\circ$, it follows by assumption that $x \to y = {\frown}(x \circ {\frown}y)$. Then $\neg x = {\frown}(x \wedge {\frown}0) = {\frown}x$. It follows that $\neg\neg x = {\frown}{\frown}x = x$, so $\mathbf{A}$ is Boolean. □

Corollary 7.4.6. *The subvariety of $\mathcal{SHRA}$ defined by the identity $x \circ y = x \wedge y$ is term-equivalent to the variety of Boolean algebras.*

8

Varieties of double Heyting algebras

The previous few chapters concerned general results for EHAs with a congruence-filter term. In this chapter, we will restrict our attention only to pure H^+-algebras and double Heyting algebras. In particular, we will investigate the lattice of subvarieties of $\mathcal{H}^+$ and $\mathcal{DH}$. We begin by looking at small subvarieties. The smallest non-trivial subvariety is easily seen to be the variety of Boolean algebras. We prove that the variety of Boolean algebras has a unique cover in $\mathcal{L}(\mathcal{H}^+)$ and $\mathcal{L}(\mathcal{DH})$. Using the results of Chapter 5, we also prove that there are exactly two splitting algebras in each of $\mathcal{DH}$ and $\mathcal{H}^+$. This was proved for double Heyting algebras by Wolter [93, Theorem 31], who showed that $\mathcal{L}(\mathcal{DH})$ is isomorphic to the lattice of subvarieties of a certain subvariety of tense algebras. The corresponding results for that subvariety were proved by Kracht [64]. Our proof is different, and also extends the result to H^+-algebras, which cannot be derived from the work in [93] and [64].

We actually characterise finite non-splitting algebras in subvarieties of $\mathcal{DH}$ and $\mathcal{H}^+$ satisfying certain closure conditions. This applies to $\mathcal{RDP}$, the variety of regular double p-algebras. If $\mathcal{RDP}$ is generated by its finite members, then all of its splitting algebras are finite, in which case the characterisation is complete. The question of whether $\mathcal{RDP}$ is generated by its finite members is still open. In the last section of this chapter we look at some other subvarieties of $\mathcal{DH}$ that satisfy the closure conditions. The variety of regular double p-algebras is generalised by classes defined by a forbidden configuration in the dual space—$\mathcal{RDP}$ is obtained by forbidding the 3-element chain. Forbidden configurations for Heyting algebras were studied by Ball and Pultr [3]. We do not prove much in the last section, but we use a result of Ball and Pultr to give infinitely many subvarieties of $\mathcal{DH}$ and $\mathcal{H}^+$ that the general splitting result applies to. This includes some examples that apply for $\mathcal{DH}$ but not $\mathcal{H}^+$. Unless otherwise stated, the results of this chapter are entirely the original work of the author. The results are not yet published, but are included in a manuscript under preparation by Davey, Kowalski, and the author.

8.1 Small subvarieties

Since every non-trivial double Heyting algebra has $\{0, 1\}$ as a subuniverse, the variety of Boolean algebras is the minimum non-trivial subvariety of $\mathcal{DH}$. The same thing applies to the variety of H$^+$-algebras as well. To find other small subvarieties, the next obvious candidate is the variety generated by the 3-element chain. Note that by Theorem 1.6.4, up to term-equivalence, Var($\mathbf{3}$) is the same whether treated as a variety of double Heyting algebras, a variety of H$^+$-algebras, or a variety of (regular) double p-algebras. We will start this section by using the duality of Chapter 2 to characterise, in terms of the dual space, the double Heyting algebras that have a subalgebra isomorphic to $\mathbf{3}$. The next lemma shows that it suffices to do so for H$^+$-algebras. Recall that $\mathbf{A} \leq \mathbf{B}$ means $\mathbf{A} \in \mathsf{IS}(\mathbf{B})$ and that $\mathbf{A}^\flat$ denotes the H$^+$-algebra reduct of a double Heyting algebra $\mathbf{A}$.

Lemma 8.1.1. *Let* $\mathbf{A}$ *be a double Heyting algebra. Then* $\mathbf{3} \leq \mathbf{A}$ *if and only if* $\mathbf{3} \leq \mathbf{A}^\flat$.

Proof. If $\mathbf{3}$ embeds into $\mathbf{A}$, then it is obvious that $\mathbf{3}$ embeds into $\mathbf{A}^\flat$. For the converse, it is easily checked that, for all $x \in A$, we have $x \dot{-} 1 = 0$, $0 \dot{-} x = 0$, and $x \dot{-} 0 = x$. Consequently, if $\{0, x, 1\}$ is a subuniverse of $\mathbf{A}^\flat$, then it is closed under $\dot{-}$ and therefore it is a subuniverse of $\mathbf{A}$. □

Since finite products of H$^+$-algebras correspond to disjoint unions of ordered sets in the Priestley dual, what follows is a consequence of Theorem 1.4.6.

Proposition 8.1.2. *Let X be a finite ordered set. Then, as an H$^+$-algebra or a double Heyting algebra,* $\mathcal{U}(X)$ *is simple if and only if X is connected.*

Definition 8.1.3. Let X be an H$^+$-space. If $x \in \min(X) \cap \max(X)$, then we will call x *order-isolated*.

Recall that under the duality, if $\mathbf{A}$ and $\mathbf{B}$ are H$^+$-algebras, then an embedding $h\colon \mathbf{A} \to \mathbf{B}$ corresponds to a surjective H$^+$ morphism $\varphi\colon \mathcal{F}_p(\mathbf{B}) \to \mathcal{F}_p(\mathbf{A})$.

Proposition 8.1.4. *Let X be an H$^+$-space. There exists a surjective H$^+$-morphism* $\varphi\colon X \to \mathbf{2}$ *if and only if X has no order-isolated elements.*

Proof. If $\min(X) \cap \max(X) = \varnothing$, then, since $\min(X)$ and $\max(X)$ are closed subsets of X, there exists a clopen upset U such that $\max(X) \subseteq U$ and $\min(X) \cap U = \varnothing$. It is then easily verified that the set $\{\varnothing, U, X\}$ is an H$^+$-subuniverse of $\mathcal{U}^\mathcal{T}(X)$. Conversely, let $x \in X$, assume that x is order-isolated, and let φ be an H$^+$ morphism on X. By Lemma 2.3.1, it follows that $\varphi(x)$ is both minimal and maximal. Since $\mathbf{2}$ has no such elements, the codomain of φ cannot be $\mathbf{2}$. □

Corollary 8.1.5. *Let X be a double Heyting space. Then there exists a surjective double Heyting morphism $\varphi\colon X \to \mathbf{2}$ if and only if X has no order-isolated elements.*

This is not enough to show that every non-trivial and non-Boolean subvariety of double Heyting algebras contains the 3-element chain. The H^+-space depicted in Figure 8.1 is the dual of a subdirectly irreducible H^+-algebra, and it has an order-isolated element, so the algebra has no subalgebra isomorphic to $\mathbf{3}$. Yet, as we will see shortly, the variety it generates contains $\mathbf{3}$. On the other hand, it is true that $\mathbf{3}$ embeds into every *finite* non-Boolean subdirectly irreducible double Heyting algebra. Indeed, by Proposition 8.1.2, if $\mathcal{U}(X)$ is a finite double Heyting algebra, then X is connected, so it cannot have any order-isolated elements unless $|X| = 1$.

Corollary 8.1.6. *If $\mathbf{A}$ is a finite non-Boolean subdirectly irreducible double Heyting algebra or H^+-algebra, then $\mathbf{3} \leq \mathbf{A}$.*

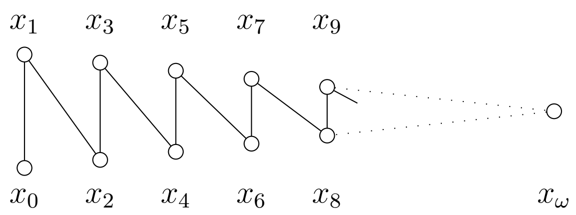


Figure 8.1: If X is the Priestley space depicted above, then $\mathcal{U}^{\mathcal{T}}(X)$ is subdirectly irreducible. The congruence lattice is a 3-element chain.

To prove that every non-trivial and non-Boolean subvariety of double Heyting algebras contains the 3-element chain, the next lemma will be useful. For convenience, let $qx = {\sim}\neg x$.

Lemma 8.1.7. *Let X be an H^+-space and let U be a clopen upset in X. If $U \neq \varnothing$, then $d^n q^{n+1} U \neq \varnothing$, for all $n \in \omega$.*

Proof. Suppose that $d^n q^{n+1} U = \varnothing$. This means that $({\downarrow}{\uparrow})^n (X \backslash ({\uparrow}{\downarrow})^{n+1} U) = X$. Then, for each $u \in U$, there exists $y \in X \backslash ({\uparrow}{\downarrow})^{n+1} U$ such that $u \in ({\downarrow}{\uparrow})^n y$. But then $y \in ({\downarrow}{\uparrow})^n u \subseteq {\uparrow}({\downarrow}{\uparrow})^n {\downarrow} u \subseteq ({\uparrow}{\downarrow})^{n+1} U$, a contradiction. □

Theorem 8.1.8. *Let $\mathbf{A}$ be an H^+-algebra. If $\mathbf{A}$ is not Boolean, then $\mathbf{3} \in \mathrm{Var}(\mathbf{A})$. More precisely, if $\mathbf{A}$ is non-Boolean and subdirectly irreducible, then there exists a congruence $\alpha \in \mathrm{Con}(\mathbf{A}^\omega)$ such that $\mathbf{3} \leq \mathbf{A}^\omega/\alpha$.*

Proof. Let X be the Priestley dual of $\mathbf{A}$ and assume that $\mathbf{A}$ is non-Boolean and subdirectly irreducible. If X has no order-isolated elements, then we are covered by Proposition 8.1.4. So, assume that X has at least one order-isolated element. Recall that $\min_X(U) = \min(X) \cap U$ and $\max_X(U) = \max(X) \cap U$, for all $U \subseteq X$. If $X = \min(X)$, then $\mathbf{A}$ is Boolean. So $X \setminus \min(X)$ is non-empty, and since $\min(X)$ is closed, there exists a non-empty clopen upset $U \subseteq X$ such that $\min_X(U) = \varnothing$. Then U cannot contain any order-isolated elements. But X does, so we must have $q^iU = (\uparrow\downarrow)^iU \neq X$, for all $i \in \omega$. Additionally, if there exists $i \in \omega$ such that $q^iU = q^{i+1}U$, then q^iU is complemented, and in a subdirectly irreducible H^+-algebra this only occurs if $q^iU = \varnothing$ or $q^iU = X$. We have already seen that $q^iU \neq X$, for all $i \in \omega$. Moreover, we have $qU = 0$ if and only if $U = 0$, and so, by induction, the former case does not occur either. Therefore, the members of $\langle q^iU\rangle_{i\in\omega}$ are pairwise distinct. Let $U_i = q^iU$.

Since $\max(X)$ is closed, $\max_X(U_i)$ is also closed. Hence, for each $i \in \omega$, there is a non-empty clopen upset V_i such that $\max_X(U_i) \subseteq V_i$ and $\min_X(V_i) = \varnothing$. Let $M_i = V_i \cap U_i$, and observe that $\max_X(M_i) = \max_X(U_i)$ and $\min_X(M_i) = \varnothing$. Because they share their maximal elements, we have $\downarrow M_i = \downarrow U_i$ and it follows that $\neg M_i = \neg U_i$. Moreover, since $\min_X(M_i) = \varnothing$, we have $\uparrow(X \setminus M_i) = X$ and therefore ${\sim} M_i = X$.

Now let $\mathbf{H} = \mathbf{A}^\omega$. Denote the tuple $\langle M_i\rangle_{i\in\omega}$ by M, and let α be the congruence

$$\alpha = \mathrm{Cg}^{\mathbf{H}}(\neg M, 0).$$

In any H^+-algebra, $\neg x = 1$ if and only if $x = 0$, so we then have

$$\alpha = \mathrm{Cg}^{\mathbf{H}}(\neg\neg M, 1) = \mathrm{Cg}^{\mathbf{H}}(\neg\neg\langle U_i\rangle_{i\in\omega}, 1).$$

To see that α is not the full congruence on $\mathbf{H}$, we will suppose that it is. Then there exists $n \in \omega$ such that $d^n\neg\neg\langle U_i\rangle_{i\in\omega} = 0$. Since d is order-preserving and $\neg\neg x \geq x$, we have $d^n\langle U_i\rangle_{i\in\omega} = 0$. In other words, for each $i \in \omega$, we have $d^nU_i = d^nq^iU = \varnothing$. But by Lemma 8.1.7 this is impossible. Hence, $\mathbf{H}/\alpha$ is a non-trivial algebra.

We finish the proof by showing that $\mathbf{3} \leq \mathbf{H}/\alpha$. Since ${\sim} M_i = X$, for all $i \in \omega$, it follows that ${\sim} M = 1$ in $\mathbf{H}$, so ${\sim} M/\alpha = 1/\alpha$. By definition of α, we have $\neg M/\alpha = 0/\alpha$. These two facts combined with the fact that $\mathbf{H}/\alpha$ is non-trivial imply that $M/\alpha \notin \{0/\alpha, 1/\alpha\}$. We thus conclude that $\{0/\alpha, M/\alpha, 1/\alpha\}$ is the underlying set of a subalgebra of $\mathbf{H}/\alpha$ isomorphic to $\mathbf{3}$. □

A similar argument also applies to double Heyting algebras, but assuming ignorance of the proof, we can still prove the analogous result as a direct corollary. Let $\mathbf{A}$ be a non-Boolean subdirectly irreducible double Heyting algebra. By the previous result, there exists a congruence α on $(\mathbf{A}^\flat)^\omega$ such that $\mathbf{3} \leq (\mathbf{A}^\flat)^\omega/\alpha$. But since the operations $\to$ and $\dot{-}$ depend only on the underlying lattice, it follows that $(\mathbf{A}^\flat)^\omega = (\mathbf{A}^\omega)^\flat$. By Theorem 1.5.6, we have $\mathrm{Con}(\mathbf{A}^\omega) = \mathrm{Con}((\mathbf{A}^\omega)^\flat)$, so α is a congruence on $\mathbf{A}^\omega$. But we also have $(\mathbf{A}^\omega/\alpha)^\flat = (\mathbf{A}^\omega)^\flat/\alpha = (\mathbf{A}^\flat)^\omega/\alpha$. So, by Lemma 8.1.1, it follows that $\mathbf{3} \leq \mathbf{A}^\omega/\alpha$, as claimed. The next two results follow by observing that by the previous result, the only subvarieties not containing $\mathbf{3}$ are the trivial subvariety and the variety of Boolean algebras.

Corollary 8.1.9. *In $\mathcal{L}(\mathcal{H}^+)$, the variety* $\mathrm{Var}(\mathbf{3})$ *is completely join-irreducible and covers the variety* $\mathrm{Var}(\mathbf{2})$. *Hence,* $\mathbf{3}$ *is a splitting algebra in* $\mathcal{H}^+$.

Corollary 8.1.10 (Wolter [93])**.** *In $\mathcal{L}(\mathcal{DH})$, the variety* $\mathrm{Var}(\mathbf{3})$ *is completely join-irreducible and covers the variety* $\mathrm{Var}(\mathbf{2})$. *Hence,* $\mathbf{3}$ *is a splitting algebra in* $\mathcal{DH}$.

We have started investigating subvarieties covering $\mathrm{Var}(\mathbf{3})$ in $\mathcal{L}(\mathcal{DH})$. No clear patterns have emerged, but a computer search has yielded a handful of covers. We revisit this in Chapter 9.

Open Problem 12. Are there infinitely many covers of $\mathrm{Var}(\mathbf{3})$ in $\mathcal{L}(\mathcal{DH})$? Characterise them. What about $\mathcal{H}^+$ and $\mathcal{RDP}$? A different approach using algebras of incidence structures and fences may work for $\mathcal{RDP}$.

8.2 Splitting algebras

In Section 5.5, we characterised, non-constructively, the finite non-splitting algebras in certain varieties of EHAs. In particular, the characterisation applies to double Heyting algebras and H$^+$-algebras. We will begin this section by providing a sufficient condition to imply the second condition of Lemma 5.5.3 in this restricted setting. By the end of this section we will have proved that the only splitting algebras in $\mathcal{DH}$ and $\mathcal{H}^+$ are the 2-element and 3-element chains. Recall that $dx = \neg{\sim}x$, and recall by Corollary 1.4.6 that every finite subdirectly irreducible double Heyting algebra and H$^+$-algebra is simple.

Lemma 8.2.1. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be finite simple H$^+$-algebras or double Heyting algebras. Then* $\mathbf{A} \in \mathrm{Var}(\mathbf{B})$ *if and only if* $\mathbf{A} \leq \mathbf{B}$.

Proof. Since $\mathbf{A}$ and $\mathbf{B}$ are both finite simple algebras, by Jónsson's Lemma, we have $\mathbf{A} \in \mathrm{Var}(\mathbf{B})$ if and only if $\mathbf{A} \in \mathsf{HS}(\mathbf{B})$. Both $\mathcal{DH}$ and $\mathcal{H}^+$ have the congruence extension property, so every non-trivial algebra in $\mathsf{HS}(\mathbf{B})$ is in $\mathsf{IS}(\mathbf{B})$. □

Therefore, if $\mathcal{V}$ is a variety of H$^+$-algebras or double Heyting algebras, condition (2) of Lemma 5.5.3 is implied by

$$(\forall i \in \omega)(\exists \mathbf{B}_i \in \mathcal{V})\ \mathbf{B}_i \text{ is simple, } \mathbf{A} \not\leq \mathbf{B}_i, \text{ and } \mathbf{B}_i \nvDash d^i\Delta_{\mathbf{A}} = 0. \qquad (\dagger)$$

For convenience, in this paragraph we will speak only of H$^+$-algebras and take note that everything we say also applies to double Heyting algebras. Let $\mathcal{V}$ be a variety of H$^+$-algebras and let $\mathbf{A} \in \mathcal{V}$. From Proposition 8.1.2, if $\mathbf{A}$ is finite, then $\mathbf{A}$ is simple if and only if its Priestley dual is connected. Recall that a double-pointed ordered set is a finite ordered set with at least two elements, equipped with two nullary operations α and β specifying an arbitrary minimal and maximal element respectively. The operation $\searrow$ from Section 2.4 clearly preserves connectedness, so an algebra of the form $\mathcal{U}(\mathbf{X} \searrow \mathbf{Y})$ will be simple if and only if $\mathbf{X}$ and $\mathbf{Y}$ are connected double-pointed ordered sets. We will use this to prove that if $\mathbf{A}$ is a finite simple algebra and $|A| > 2$, then, for all $i \in \omega$, there exists a simple algebra $\mathbf{B}_i$ such that $\mathbf{A} \not\leq \mathbf{B}_i$.

Definition 8.2.2. Let $\mathbf{X}$ be a double-pointed ordered set. For each $i \geq 1$, let $X_i = X \times \{i\}$. Now let $\mathbf{X}_i$ be the double-pointed ordered set with underlying set X_i, with the order defined by $(x, i) \leq (y, i)$ if and only if $x \leq y$, and let $\alpha^{\mathbf{X}_i} = (\alpha, i)$ and $\beta^{\mathbf{X}_i} = (\beta, i)$. For each $n \geq 1$, let

$$\mathbf{X}^{(n)} = \mathbf{X}_1 \searrow \mathbf{X}_2 \searrow \dots \searrow \mathbf{X}_{n-1} \searrow \mathbf{X}_n.$$

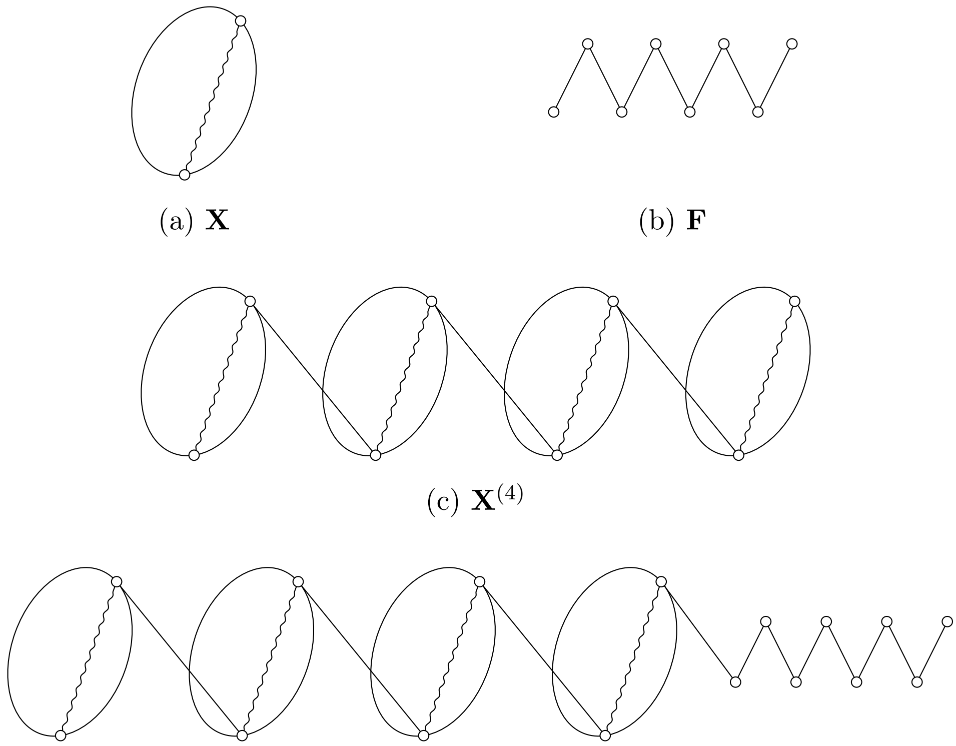
Note that $\alpha^{\mathbf{X}^{(n)}} = (\alpha, 1)$ and $\beta^{\mathbf{X}^{(n)}} = (\beta, n)$. See Figure 8.2c for an illustration. For two ordered sets X and Y, we will say that X *never maps onto* Y if there is no surjective H$^+$-morphism $\varphi\colon X \to Y$.

If X never maps onto Y, then in the dual this means that, as H$^+$-algebras, we have $\mathcal{U}(Y) \not\leq \mathcal{U}(X)$. This implies that $\mathcal{U}(Y) \not\leq \mathcal{U}(X)$ when treating them as double Heyting algebras. Thus, we only consider H$^+$-morphisms in what follows.

Proposition 8.2.3. *Let* $\mathbf{X}$ *be a finite double-pointed ordered set and let* $\mathbf{F}$ *be a fence with a down-tail. Assume that* $\mathbf{X}$ *is not a fence and that* $|F| > |X|$*. Then, for all* $i \geq 1$*, the ordered set* $\mathbf{X}^{(i)} \searrow \mathbf{F}$ *never maps onto* $\mathbf{X}$.

Proof. Because $|F| > |X|$, by the pigeonhole principle, if $\varphi\colon \mathbf{X}^{(i)} \searrow \mathbf{F} \to \mathbf{X}$ is an H$^+$-morphism, then it is not one-to-one when restricted to F. Hence, by Corollary 2.4.11, $\varphi(\mathbf{X}^{(i)} \searrow \mathbf{F})$ is a fence. Since $\mathbf{X}$ is not a fence, φ is not surjective. □

This supplies us with our candidate algebras for condition (†), provided that the dual of the algebra is not a fence. We require a special argument otherwise. If $\mathbf{X}$ is a fence that only has down-tails, then we can choose a large enough fence $\mathbf{F}$ with one up-tail. Then by Lemma 2.4.5, for all $i \geq 1$, if φ is a surjective H$^+$-morphism from $\varphi(\mathbf{X}^{(i)} \searrow \mathbf{F})$ to $\mathbf{X}$, then there is an up-tail in $\mathbf{X}$. But $\mathbf{X}$ has none, so Proposition 8.2.3 holds in this case as well. Similarly, if $\mathbf{X}$ is a fence with no



(a) $\mathbf{X}$ (b) $\mathbf{F}$

(c) $\mathbf{X}^{(4)}$

(d) $\mathbf{X}^{(4)} \searrow \mathbf{F}$

Figure 8.2: Let $\mathbf{X}$ and $\mathbf{F}$ be the ordered sets as given. To avoid clutter, the labels for α and β are not included. Note that here we have assumed that $\alpha^{\mathbf{X}} \leq \beta^{\mathbf{X}}$.

down-tails, we can choose $\mathbf{F}$ so that it only has down-tails, and by Lemma 2.3.4, the result still holds. If $\mathbf{X}$ is a two-element fence, or in other words, a two-element chain, then $\mathcal{U}(\mathbf{X}) \cong \mathbf{3}$ which we have already seen is a splitting algebra. Thus, the only case that remains is if $\mathbf{X}$ is a fence with at least 3 elements, exactly one up-tail, and exactly one down-tail.

Proposition 8.2.4. *Let* $\mathbf{X}$ *be a fence and assume* $|X| > 2$. *Then there is a fence* $\mathbf{F}$ *such that, for all* $i \geq 1$, *the ordered set* $\mathbf{X}^{(i)} \searrow \mathbf{F}$ *never maps onto* $\mathbf{X}$.

Proof. We just discussed the case that $\mathbf{X}$ has no down-tails or no up-tails. So assume that $\mathbf{X}$ has one up-tail and one down-tail. Note that this implies that $|X| \neq 3$. The elements of $\mathbf{X}$ have their order given by

$$x_1 < x_2 > x_3 < \cdots > x_{n-1} < x_n.$$

Let $\mathbf{F}$ be a fence with $|X| + 1$ elements, with the order given by

$$f_0 > f_1 < f_2 > f_3 < \cdots > f_{n-1} < f_n,$$

and let $\alpha^{\mathbf{F}} = f_1$, with $\beta^{\mathbf{F}}$ left arbitrary. Let $n \in \omega$ and let φ be a morphism from $\mathbf{X}^{(n)} \searrow \mathbf{F}$ to $\mathbf{X}$. Suppose, by way of contradiction, that $\varphi(\mathbf{X}^{(n)} \searrow \mathbf{F}) = \mathbf{X}$.

The pair (f_{n-1}, f_n) is an up-tail in $\mathbf{X}^{(i)} \searrow \mathbf{F}$, so Lemma 2.4.4 tells us that $(\varphi(f_{n-1}), \varphi(f_n))$ is an up-tail in $\mathbf{X}$. There is exactly one up-tail in $\mathbf{X}$, namely (x_{n-1}, x_n), so $\varphi(f_{n-1}) = x_{n-1}$ and $\varphi(f_n) = x_n$. We will now prove inductively that $\varphi(f_k) = x_k$, for all $i \geq 1$. Let $k > 1$ and assume that $\varphi(f_i) = x_i$, for all $i \geq k$. We will show that $\varphi(f_{k-1}) = x_{k-1}$. If f_k is minimal, then, since $\varphi(f_k) = x_k$, we have $\varphi(\uparrow f_k) = \uparrow\varphi(f_k) = \uparrow x_k = \{x_{k-1}, x_k, x_{k+1}\}$, and because $\varphi(f_{k+1}) = x_{k+1}$, we must have $\varphi(f_{k-1}) = x_{k-1}$. By Lemma 2.4.4, a dual argument holds if f_k is maximal. Thus, for all $i \geq 1$, we have $\varphi(f_i) = x_i$. Since (f_0, f_1) is an up-tail in $\mathbf{X}^{(i)} \searrow \mathbf{F}$, we must have that $(\varphi(f_0), \varphi(f_1))$ is an up-tail in $\mathbf{X}$. But $\varphi(f_1) = x_1$, and x_1 is certainly not part of any up-tail in $\mathbf{X}$, a contradiction. $\square$

Corollary 8.2.5. *Let* $\mathbf{A}$ *be a finite simple* H^+*-algebra and let* $\mathbf{X} = \mathcal{F}_p(\mathbf{A})$*, with* $\alpha^{\mathbf{X}}$ *and* $\beta^{\mathbf{X}}$ *chosen arbitrarily. Then there exists a fence* $\mathbf{F}$ *such that, for all* $i \in \omega$*, we have* $\mathbf{A} \not\leq \mathcal{U}(\mathbf{X}^{(i)} \searrow \mathbf{F})$*.*

The other part of condition (†) is evaluating the term-diagram. For this, the size and type of the fence is not important. In fact, assuming it is a fence is not even necessary. The following lemmas will aid in the calculation.

Lemma 8.2.6. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be finite connected double-pointed ordered sets. Let* U *and* V *be upsets in* $\mathbf{X}$ *and then, for each* $* \in \{\vee, \wedge, \to, \dot{-}\}$*, let* $U \mathring{*} V$ *be shorthand for* $U *^{\mathcal{U}(X)} V$*, and similarly for* $\mathring{\sim} U$*. Then, when evaluated in* $\mathcal{U}(\mathbf{X} \searrow \mathbf{Y})$*, for each* $* \in \{\vee, \wedge, \dot{-}\}$*, we have*

$$U * V = U \mathring{*} V,$$

for $\sim$ *we have*

$$\sim U = \mathring{\sim} U \cup Y \cup \{\beta^{\mathbf{X}}\},$$

and for $\to$ *we have*

$$U \to V = \begin{cases} (U \mathring{\to} V) \cup Y & \text{if } \beta^{\mathbf{X}} \notin U \backslash V, \\ (U \mathring{\to} V) \cup Y \backslash \{\alpha^{\mathbf{Y}}\} & \text{otherwise.} \end{cases}$$

Proof. First note that U and V are also upsets in $\mathbf{X} \searrow \mathbf{Y}$. Let $\Uparrow$ and $\Downarrow$ denote the operations $\uparrow$ and $\downarrow$ with respect to the order on $\mathbf{X}$. Recall by Lemma 2.2.5 that the operations listed above are given by:

$$U \mathring{\vee} V = U \cup V, \qquad U \mathring{\wedge} V = U \cap V,$$
$$U \mathring{\to} V = X \backslash {\Downarrow}(U \backslash V), \qquad U \mathring{\dot{-}} V = {\Uparrow}(U \backslash V),$$
$$\mathring{\sim} U = {\Uparrow}(X \backslash U).$$

The calculations for the lattice operations are trivial. For $\sim$, in $\mathcal{U}(\mathbf{X} \searrow \mathbf{Y})$ we have

$$\sim U = \uparrow\big[(X \cup Y)\backslash U\big] = \uparrow\big[X\backslash U \cup Y\backslash U\big] = \uparrow(X\backslash U) \cup \uparrow Y.$$

Since $\uparrow(X\backslash U)$ and Y are disjoint, we have $\uparrow(X\backslash U) = \Uparrow(X\backslash U) = \mathring{\sim}U$, and by construction, we have $\uparrow Y = Y \cup \{\beta^{\mathbf{X}}\}$. Thus $\sim U = \mathring{\sim}U \cup Y \cup \{\beta^{\mathbf{X}}\}$. For $\dot{-}$, we have $U \dot{-} V = \uparrow(U\backslash V)$. Since $U, V \subseteq X$, we have that $\uparrow(U\backslash V)$ and Y are disjoint. So $\uparrow(U\backslash V) = \Uparrow(U\backslash V)$, which proves the claim. For $\to$, we have

$$\begin{aligned} U \to V &= (X \cup Y)\backslash\downarrow(U\backslash V) \\ &= \big[X\backslash\downarrow(U\backslash V)\big] \cup \big[Y\backslash\downarrow(U\backslash V)\big] \\ &= U \mathbin{\mathring{\to}} V \cup \big[Y\backslash\downarrow(U\backslash V)\big]. \end{aligned}$$

If $\beta^{\mathbf{X}} \notin U\backslash V$, then $Y \cap \downarrow(U\backslash V) = \varnothing$, and otherwise, $Y \cap \downarrow(U\backslash V) = \{\alpha^{\mathbf{Y}}\}$. So,

$$Y\backslash\downarrow(U\backslash V) = \begin{cases} Y & \text{if } \beta^{\mathbf{X}} \notin U\backslash V, \\ Y\backslash\{\alpha^{\mathbf{Y}}\} & \text{otherwise,} \end{cases}$$

completing the proof. $\square$

Lemma 8.2.7. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be finite connected double-pointed ordered sets. Let* U *and* V *be upsets in* $\mathbf{X}$ *and then, for each* $* \in \{\vee, \wedge, \to, \dot{-}\}$, *let* $U \mathbin{\mathring{*}} V$ *be shorthand for* $U *^{\mathcal{U}(X)} V$, *and similarly for* $\mathring{\sim}U$. *Then, when evaluated in* $\mathcal{U}(\mathbf{X} \searrow \mathbf{Y})$, *for each* $* \in \{\vee, \wedge, \dot{-}\}$, *we have*

$$(U * V) \leftrightarrow (U \mathbin{\mathring{*}} V) = X \cup Y,$$

for $\sim$ *we have*

$$\sim U \leftrightarrow \mathring{\sim}U = \begin{cases} X & \textit{if } \beta^{\mathbf{X}} \in \mathring{\sim}U, \\ X\backslash\downarrow\beta^{\mathbf{X}} & \textit{otherwise,} \end{cases},$$

and for $\to$ *we have*

$$(U \to V) \leftrightarrow (U \mathbin{\mathring{\to}} V) = X.$$

Proof. The first part holds because $U * V = U \mathbin{\mathring{*}} V$ whenever $* \in \{\vee, \wedge, \dot{-}\}$. For $\sim$, we have $\mathring{\sim}U \subseteq \sim U$, so

$$\sim U \leftrightarrow \mathring{\sim}U = \sim U \to \mathring{\sim}U = (X \cup Y)\backslash\downarrow(\sim U\backslash\mathring{\sim}U).$$

Now,

$$\downarrow(\sim U\backslash\mathring{\sim}U) = \begin{cases} Y & \text{if } \beta^{\mathbf{X}} \in \mathring{\sim}U, \\ Y \cup \{\beta^{\mathbf{X}}\} & \text{otherwise.} \end{cases}$$

Hence,

$$
\begin{aligned}
{\sim}U \rightarrow \mathring{\sim}U &= \begin{cases} (X \cup Y)\backslash Y & \text{if } \beta^{\mathbf{X}} \in \mathring{\sim}U, \\ (X \cup Y)\backslash (Y \cup {\downarrow}\beta^{\mathbf{X}}) & \text{otherwise,} \end{cases} \\
&= \begin{cases} X & \text{if } \beta^{\mathbf{X}} \in \mathring{\sim}U, \\ X\backslash {\downarrow}\beta^{\mathbf{X}} & \text{otherwise,} \end{cases}
\end{aligned}
$$

as required. For $\rightarrow$, we have $U \mathring{\rightarrow} V \subseteq U \rightarrow V$, so

$$(U \rightarrow V) \leftrightarrow (U \mathring{\rightarrow} V) = (U \rightarrow V) \rightarrow (U \mathring{\rightarrow} V).$$

First observe that

$$(U \rightarrow V)\backslash(U \mathring{\rightarrow} V) = \begin{cases} Y & \text{if } \beta^{\mathbf{X}} \notin U\backslash V, \\ Y\backslash\{\alpha^{\mathbf{Y}}\} & \text{otherwise,} \end{cases}$$

and in either case we have ${\downarrow}[(U \rightarrow V)\backslash(U \mathring{\rightarrow} V)] = Y$. Hence,

$$(U \rightarrow V) \rightarrow (U \mathring{\rightarrow} V) = (X \cup Y)\backslash{\downarrow}[(U \rightarrow V)\backslash(U \mathring{\rightarrow} V)] = (X \cup Y)\backslash Y = X,$$

as claimed. □

The next lemma is now immediate.

Lemma 8.2.8. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be double-pointed ordered sets. Let* U *and* V *be upsets in* $\mathbf{X}$ *and then, for each* $* \in \{\vee, \wedge, \rightarrow, \dot{-}\}$, *let* $U \mathring{*} V$ *be shorthand for* $U *^{\mathcal{U}(X)} V$, *and similarly for* $\mathring{\sim}U$. *Let* $\chi(U, V)$ *denote the element of* $\mathcal{U}(\mathbf{X} \searrow \mathbf{Y})$ *given by*

$$
\begin{aligned}
\chi(U, V) =& [(U \mathring{\wedge} V) \leftrightarrow (U \wedge V)] \wedge [(U \mathring{\vee} V) \leftrightarrow (U \vee V)] \\
& \wedge [(U \mathring{\rightarrow} V) \leftrightarrow (U \rightarrow V)] \wedge [(U \mathring{\dot{-}} V) \leftrightarrow (U \dot{-} V)].
\end{aligned}
$$

Then $\chi(U, V) = X$. *Similarly, if* $\chi^{+}(U, V)$ *is given by*

$$
\begin{aligned}
\chi^{+}(U, V) =& [(U \mathring{\wedge} V) \leftrightarrow (U \wedge V)] \wedge [(U \mathring{\vee} V) \leftrightarrow (U \vee V)] \\
& \wedge [(U \mathring{\rightarrow} V) \leftrightarrow (U \rightarrow V)] \wedge [(\mathring{\sim}U \leftrightarrow {\sim}U)],
\end{aligned}
$$

then

$$\chi^{+}(U, V) = \begin{cases} X & \textit{if } \beta^{\mathbf{X}} \in \mathring{\sim}U, \\ X\backslash{\downarrow}\beta^{\mathbf{X}} & \textit{otherwise.} \end{cases}$$

With these calculations established, we can now evaluate the term-diagram. We will include one extra assumption, namely that $\alpha^{\mathbf{X}} \leq \beta^{\mathbf{X}}$. This is not a problematic assumption, since we can always find such a pair of elements in any finite connected ordered set with two or more elements.

Definition 8.2.9. Let $\mathbf{X}$ be a finite connected double-pointed ordered set, assume that $\alpha^{\mathbf{X}} \leq \beta^{\mathbf{X}}$, and let $\mathbf{A} = \mathcal{U}(X)$. For each $i \in \omega$ and each $a \in A$, let $a_i = a \times \{i\}$, and then let $U_n(a)$ denote the element of $\mathbf{X}^{(n)}$ given by

$$U_n(a) = \bigcup_{i \leq n} a_i.$$

Refer to Figure 8.3 for an illustration.

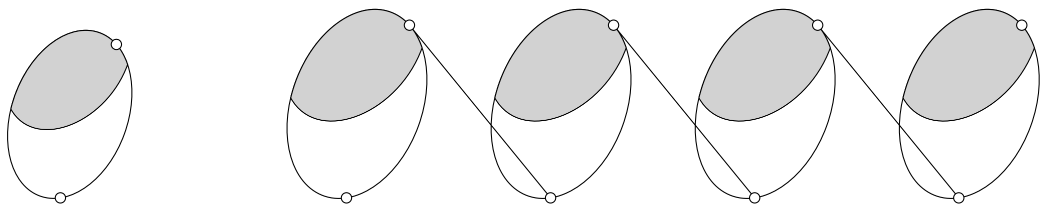


Figure 8.3: If $\mathbf{X}$ is the ordered set depicted on the left and a is the shaded part, then $U_4(a)$ is the shaded area on the right hand diagram.

By assuming $\alpha^{\mathbf{X}} \leq \beta^{\mathbf{X}}$, we ensure that $U(a)$ is an upset, for all $a \in A$. We will maintain this assumption for all double-pointed ordered sets in the remainder of this section.

Lemma 8.2.10. *Let $\mathbf{X}$ be a finite connected double-pointed ordered set and let $n \in \omega$. The map $U_n \colon \mathcal{U}(\mathbf{X}) \to \mathcal{U}(\mathbf{X}^{(n)})$ given by $a \mapsto U_n(a)$ is a double Heyting algebra homomorphism.*

Proof. We show that the map $h \colon \mathbf{X}^{(n)} \to \mathbf{X}$ given by $(x, i) \mapsto x$ is a double Heyting morphism whose dual is U. Demanding that $\alpha^{\mathbf{X}} \leq \beta^{\mathbf{X}}$ ensures that ${\downarrow}h(x) = h({\downarrow}x)$ and ${\uparrow}h(x) = h({\uparrow}x)$. Moreover, for each $a \in \mathcal{U}(X)$, we have

$$h^{-1}(a) = \{(x,i) \in \mathbf{X}^{(n)} \mid h((x,i)) \in a\} = \{(x,i) \in \mathbf{X}^{(n)} \mid x \in a\} = U_n(a),$$

which proves that h is the dual of U. □

It follows immediately that the map U is also an H$^+$-algebra homomorphism. Now let $\mathbf{A}$ be a finite non-Boolean simple H$^+$-algebra and let $\mathbf{X}$ be a double-pointed ordered set such that $\mathbf{A} \cong \mathcal{U}(\mathbf{X})$. From Corollary 8.2.5, there exists a finite connected double-pointed ordered set $\mathbf{F}$ such that $\mathbf{A} \not\leq \mathcal{U}(\mathbf{X}^{(i)} \searrow \mathbf{F})$, for all $i \geq 1$. For each $i \in \omega$, let $\mathbf{C}_i = \mathcal{U}(\mathbf{X}^{(i+2)} \searrow \mathbf{F})$. Then $\mathbf{A} \not\leq \mathbf{C}_i$. The use of $i+2$ is necessary for Lemma 8.2.11 to work for H$^+$-algebras—for double Heyting algebras, $i+1$ would suffice. All that remains is to prove the following:

$$(\forall i \in \omega)\ \mathbf{C}_i \nvDash d^i \Delta_{\mathbf{A}} = 0.$$

Let $\Delta_{\mathbf{A}}^{\mathcal{DH}}$ denote the term-diagram of $\mathbf{A}$ as a double Heyting algebra and let $\Delta_{\mathbf{A}}^{\mathcal{H}^+}$ denote the term-diagram of $\mathbf{A}$ as an H$^+$-algebra. Then,

$$\begin{aligned}\Delta_{\mathbf{A}}^{\mathcal{DH}}(\overline{x}) = \bigwedge\{&[x_{a\wedge b} \leftrightarrow (x_a \wedge x_b)] \wedge [x_{a\vee b} \leftrightarrow (x_a \vee x_b)]\\ &\wedge [x_{a\to b} \leftrightarrow (x_a \to x_b)] \wedge [x_{a\dot{-} b} \leftrightarrow (x_a \dot{-} x_b)]\\ &\wedge [x_0 \leftrightarrow 0] \wedge [x_1 \leftrightarrow 1] \mid a, b \in A\},\\ \Delta_{\mathbf{A}}^{\mathcal{H}^+}(\overline{x}) = \bigwedge\{&[x_{a\wedge b} \leftrightarrow (x_a \wedge x_b)] \wedge [x_{a\vee b} \leftrightarrow (x_a \vee x_b)]\\ &\wedge [x_{a\to b} \leftrightarrow (x_a \to x_b)] \wedge [x_{\sim a} \leftrightarrow {\sim} x_a]\\ &\wedge [x_0 \leftrightarrow 0] \wedge [x_1 \leftrightarrow 1] \mid a, b \in A\}.\end{aligned}$$

Notice that the next lemma does not rely on any particular choice of $\mathbf{Y}$.

Lemma 8.2.11. *Let $\mathbf{X}$ and $\mathbf{Y}$ be finite connected double-pointed ordered sets and let $\mathbf{A} = \mathcal{U}(\mathbf{X})$. For each $n \in \omega$, let $\mathbf{C}_n = \mathcal{U}(\mathbf{X}^{(n+2)} \searrow \mathbf{Y})$. Then $\mathbf{C}_n \nvDash d^n\Delta_{\mathbf{A}} = 0$.*

Proof. Let $n \in \omega$ and, for convenience, let $\mathbf{Z} = \mathbf{X}^{(n+2)}$, so that $\mathbf{C}_n = \mathcal{U}(\mathbf{Z} \searrow \mathbf{Y})$. Observe that because $\uparrow\mathbf{Z} = \mathbf{Z}$ in $\mathbf{Z} \searrow \mathbf{Y}$, we have $\mathcal{U}(\mathbf{Z}) \subseteq \mathcal{U}(\mathbf{Z} \searrow \mathbf{Y}) = \mathbf{C}_n$, and so $U_{n+2}(a) \in \mathbf{C}_n$, for each $a \in A$. Henceforth, we will omit $n+2$ from the subscript of U. Map the variable x_a into $\mathbf{C}_n$ by $x_a \mapsto U(a)$. As we did earlier, for each $* \in \{\vee, \wedge, \to, \dot{-}, \sim\}$, let $\mathring{*}$ be shorthand for $*^{\mathcal{U}(\mathbf{Z})}$. Lemma 8.2.10 then tells us that $x_{a*b} = U(a * b) = U(a) \mathring{*} U(b)$ and $U({\sim}a) = \mathring{\sim} U(a)$, for all $a, b \in A$. We also have $U(0) = \varnothing$ and $U(1) = Z \cup Y$. Each $U(a)$ is a subset of $\mathbf{Z}$, so Lemma 8.2.8 applies. Define χ and χ^+ as in Lemma 8.2.8. By the definition of $\Delta_{\mathbf{A}}$, and evaluating it in $\mathcal{U}(\mathbf{Z} \searrow \mathbf{Y})$, we then obtain

$$\begin{aligned}\Delta_{\mathbf{A}}^{\mathcal{DH}}(\overline{x}) &= \bigwedge\{\chi(U(a), U(b)) \wedge \neg U(0) \wedge U(1) \mid a, b \in A\} = Z,\\ \Delta_{\mathbf{A}}^{\mathcal{H}^+}(\overline{x}) &= \bigwedge\{\chi^+(U(a), U(b)) \wedge \neg U(0) \wedge U(1) \mid a, b \in A\} = Z \backslash {\downarrow}\beta^{\mathbf{Z}},\end{aligned}$$

where the latter equality holds by choosing $U(a) = U(1)$. In each case we have $\mathbf{X}^{(n+1)} \subseteq \Delta_{\mathbf{A}}(\overline{x})$. Now write $\mathbf{W} = \mathbf{Z} \searrow \mathbf{Y}$. In $\mathbf{C}_n$, we have

$$d^n\mathbf{X}^{(n+1)} = \mathbf{W} \backslash ({\downarrow}{\uparrow})^n(\mathbf{W} \backslash \mathbf{X}^{(n+1)}) = \mathbf{W} \backslash ({\downarrow}{\uparrow})^n(\mathbf{X}_{n+2} \cup \mathbf{Y}),$$

and this is equal to $\varnothing$ if and only if $({\downarrow}{\uparrow})^n(\mathbf{X}_{n+2} \cup \mathbf{Y}) = \mathbf{W}$. But, by construction, the leftmost part $\mathbf{X}_1$ is not a subset of $({\downarrow}{\uparrow})^n(\mathbf{X}_{n+2} \cup \mathbf{Y})$. So $d^n\mathbf{X}^{(n+1)} \neq \varnothing$. Then since d is order-preserving, we have $d^n\Delta_{\mathbf{A}}(\overline{x}) \neq \varnothing$. Hence, $\mathbf{C}_n \nvDash d^n\Delta_{\mathbf{A}} = 0$. □

We will say that a variety $\mathcal{V}$ of H$^+$-algebras or double Heyting algebras *contains all fences* if $\mathcal{U}(F) \in \mathcal{V}$, for every fence F. We will say that it is *finitarily closed under* $\searrow$ provided that, for every double-pointed ordered set $\mathbf{X}$ and $\mathbf{Y}$, if $\mathcal{U}(\mathbf{X})$ and

$\mathcal{U}(\mathbf{Y})$ are in $\mathcal{V}$, then $\mathcal{U}(\mathbf{X} \searrow \mathbf{Y}) \in \mathcal{V}$ as well. Let $\mathcal{V}$ be a variety of double Heyting algebras or H$^+$-algebras, assume that $\mathcal{V}$ is finitarily closed under $\searrow$, and assume that $\mathcal{V}$ contains all fences. For every finite subdirectly irreducible algebra $\mathbf{A} \in \mathcal{V}$ such that $|A| > 3$, we now have

$$(\forall i \in \omega)(\exists \mathbf{B}_i \in \mathcal{V})\ \mathbf{B}_i \text{ is simple, } \mathbf{A} \nleq \mathbf{B}_i, \text{ and } \mathbf{B}_i \nvDash d^i \Delta_{\mathbf{A}} = 0.$$

Recall that **2** is trivially a splitting algebra, and Theorem 8.1.8 ensures that **3** is splitting. Thus, by Lemma 5.5.3, we have proved the following theorem.

Theorem 8.2.12. *Let $\mathcal{V}$ be a variety of H^+-algebras or double Heyting algebras. If $\mathcal{V}$ is finitarily closed under $\searrow$ and contains all fences, then the only finite splitting algebras in $\mathcal{V}$ are* **2** *and* **3**.

By Theorem 3.1.4 and Lemma 3.2.10 we conclude that every splitting algebra in each of $\mathcal{DH}$ and $\mathcal{H}^+$ is finite.

Corollary 8.2.13 (Wolter [93])**.** *The only splitting algebras in $\mathcal{DH}$ are* **2** *and* **3**.

Corollary 8.2.14. *The only splitting algebras in $\mathcal{H}^+$ are* **2** *and* **3**.

This highlights a vast difference between double Heyting algebras and Heyting algebras, because in the latter case, every finite subdirectly irreducible algebra is splitting. The case for splittings in the variety of regular double p-algebras is still incomplete. It certainly contains all fences and is finitarily closed under $\searrow$, but we do not know if it is generated by its finite members. Thus, it could still hold that there are infinite splitting algebras in $\mathcal{RDP}$.[1]

Corollary 8.2.15. *The only finite splitting algebras in $\mathcal{RDP}$ are* **2** *and* **3**.

Open Problem 13. Are there any infinite splitting algebras in $\mathcal{RDP}$?

8.3 Forbidden configurations

In this section, we will exhibit some varieties satisfying the hypothesis of Theorem 8.2.12.

Definition 8.3.1. Let X be an ordered set. We say that X has *length* n if the largest chain in X has $n + 1$ elements. If $\mathcal{V}$ is a variety of EHAs, then, for each $n > 1$, let $\mathrm{Forb}(\mathcal{V}, n)$ denote the class of algebras in $\mathcal{V}$ whose dual spaces have no n-element chain, i.e., those whose dual has length at most $n - 1$.

[1]Added in proof: we mentioned in a footnote on page 25 that Tomasz Kowalski has announced a proof that $\mathcal{RDP}$ is generated by its finite members, in which case it will be true that there are no infinite splitting regular double p-algebras.

Let $\mathcal{H}$ denote the variety of Heyting algebras. It is easily seen that $\mathrm{Forb}(\mathcal{H}, 2)$ is the variety of Boolean algebras. Interestingly, for each $n > 1$, it turns out that the class $\mathrm{Forb}(\mathcal{H}, n)$ is equational, defined by a single identity relative to $\mathcal{H}$ given as follows. For each $i > 1$, define the term c_i in variables $\{x_2, \ldots, x_i\}$ by

$$c_2 = \neg x_2,$$
$$c_{i+1} = x_{i+1} \to (x_i \vee c_i).$$

If $\mathbf{A}$ is a Heyting algebra, then $\mathbf{A} \in \mathrm{Forb}(\mathcal{H}, n)$ if and only if $\mathbf{A} \models x_n \vee c_n = 1$. These equations can be derived from Adams and Beazer [1, Theorem 3.17], where distributive lattices whose dual contains no n-element chain are characterised by a first-order formula in the language of bounded lattices. Hence, if $\mathcal{V}$ is a variety of EHAs, then $\mathrm{Forb}(\mathcal{V}, n)$ is a variety. For all $n > 2$, the class $\mathrm{Forb}(\mathcal{DH}, n)$ is clearly finitarily closed under $\searrow$ and contains all fences. Thus, Theorem 8.2.12 applies to each of those subvarieties. This "forbidden configuration" approach to distributive lattices was studied in greater generality by Ball and Pultr [3], who gave first-order formulas characterising distributive lattices whose dual space does not contain a given finite forest, i.e., a disjoint union of trees.

Definition 8.3.2. Let T be a finite ordered set and let $\mathrm{Forb}(\mathcal{H}, T)$ denote the class of Heyting algebras $\mathbf{A}$ for which T does not order-embed into $\mathcal{F}_p(\mathbf{A})$. A *tree* is a non-trivial finite connected ordered set such that every element has at most one lower cover. A tree must have a minimum element, which we will call the *root*.

Note that Ball and Pultr consider the duality using clopen downsets rather than clopen upsets, so trees are defined dually in [3]. In the same way that the first-order formula of Adams and Beazer becomes an equation for Heyting algebras, for a finite tree, the first-order formula by Ball and Pultr becomes an equation. Ball and Pultr proved the following remarkable result.

Theorem 8.3.3 (Ball and Pultr [3])**.** *Let T be a finite ordered set. The class* $\mathrm{Forb}(\mathcal{H}, T)$ *is a variety if and only if T is a tree. In that case,* $\mathrm{Forb}(\mathcal{H}, T)$ *is defined* (*relative to the variety of Heyting algebras*) *by a single identity $a_T = 1$, and a_T is in the language* $\{\vee, \to\}$.

Their proof is constructive, so it is possible to write down the term a_T, but we will not reproduce it here. Every chain is a tree, and in that case the method of Ball and Pultr produces the same formula as given by Adams and Beazer.

Definition 8.3.4. Let $\mathcal{V}$ be a variety of EHAs and let T be a finite ordered set. Let $\mathrm{Forb}(\mathcal{V}, T)$ denote the class of algebras $\mathbf{A}$ in $\mathcal{V}$ for which T does not order-embed into $\mathcal{F}_p(\mathbf{A})$.

It follows from Theorem 8.3.3 that if $\mathcal{V}$ is a variety of EHAs and T is a finite tree, then $\mathrm{Forb}(\mathcal{V}, T)$ is a subvariety of $\mathcal{V}$ defined relative to $\mathcal{V}$ by a single equation. We can use this to create an infinite class of double Heyting algebras satisfying the conditions of Theorem 8.2.12.

Proposition 8.3.5. *Let T be a tree and let $\mathcal{V} \in \{\mathcal{DH}, \mathcal{H}^+\}$. Then* $\mathrm{Forb}(\mathcal{V}, T)$ *is finitarily closed under $\searrow$ if and only if T has no maximal element that covers its root.*

Proof. Assume that T has no maximal element covering its root and let $\mathbf{X}$ and $\mathbf{Y}$ be double-pointed ordered sets such that T does not order-embed into either $\mathbf{X}$ or $\mathbf{Y}$. Suppose there is an order-embedding $\varphi\colon T \to \mathbf{X} \searrow \mathbf{Y}$. Let x be the root of T. If $\varphi(x) \neq \alpha^{\mathbf{Y}}$, then T embeds into in one of $\mathbf{X}$ and $\mathbf{Y}$, a contradiction. So $\varphi(x) = \alpha^{\mathbf{Y}}$, but then we must have some $y \in T$ such that $\varphi(y) = \beta^{\mathbf{X}}$, as otherwise T embeds into $\mathbf{Y}$. But this is also a contradiction, because then y is a maximal element covering the root of T. So T does not embed into $\mathbf{X} \searrow \mathbf{Y}$, and hence $\mathrm{Forb}(\mathcal{V}, T)$ is finitarily closed under $\searrow$. Conversely, assume that T has a maximal element covering the root. Let x be the root of T and let y be a maximal element covering x. Let $\mathbf{S}$ be a double-pointed ordered set such that its underlying ordered set is $T \backslash \{y\}$ and $\alpha^{\mathbf{S}} = x$. It is easy to see that T order-embeds into $\mathbf{S}^{(2)}$, so $\mathrm{Forb}(\mathcal{V}, T)$ is not finitarily closed under $\searrow$. □

It is easy to see that if T is a tree such that no maximal element covers its root, then the length of T is at least 2. Because fences have length 1, in that case, for each $\mathcal{V} \in \{\mathcal{DH}, \mathcal{H}^+\}$, the variety $\mathrm{Forb}(\mathcal{V}, T)$ contains all fences.

Corollary 8.3.6. *Let T be a tree and assume no maximal element in T covers the root. Then the only finite splitting algebras in* $\mathrm{Forb}(\mathcal{DH}, T)$ *and* $\mathrm{Forb}(\mathcal{H}^+, T)$ *are* **2** *and* **3**.

On the other hand, Theorem 8.3.3 can be dualised in the presence of $\dot{-}$, so forbidding duals of trees in double Heyting algebras is also an equational property. The next result does not apply for H^+-algebras.

Corollary 8.3.7. *Let T be a tree and assume no maximal element in T covers the root. Let T^∂ denote the order-theoretic dual of T. Then* $\mathrm{Forb}(\mathcal{DH}, T^\partial)$ *is a variety, and the only finite splitting algebras in* $\mathrm{Forb}(\mathcal{DH}, T^\partial)$ *are* **2** *and* **3**.

Open Problem 14. Are the forbidden tree varieties generated by their finite members? This generalises the same problem for regular double p-algebras, since $\mathcal{RDP}$ is term-equivalent to both $\mathrm{Forb}(\mathcal{H}^+, 3)$ and $\mathrm{Forb}(\mathcal{DH}, 3)$.

Open Problem 15. If T is a tree with a maximal element covering the root, then $\mathrm{Forb}(\mathcal{DH}, T)$ and $\mathrm{Forb}(\mathcal{H}^+, T)$ are varieties, but Theorem 8.2.12 does not apply. Does this suggest there are non-trivial splitting algebras in those classes, or is a different construction required?

9

Loose ends

In this chapter, we explore some miscellaneous problems that we have not yet investigated deeply, due to time constraints or otherwise. We offer no complete solutions, but a handful of potential strategies are mentioned. Recall that it was shown in Chapter 3 that H^+-algebras and double Heyting algebras are genuinely different structures. In the first section of this chapter we follow that up by asking, *how different are they?* In the second section, we will show how to translate the graph reconstruction conjecture—a longstanding open problem in graph theory—into a new problem concerning lattices. We have not solved the conjecture, but it may shed some light on the answer. In the last section, we return to lattices of subvarieties. A computer search yielded several covers of $\mathrm{Var}(\mathbf{3})$ in $\mathcal{L}(\mathcal{DH})$, but we have not observed any patterns among them. We will also see how the geometric results of Chapter 4 could be used to give insight into the lattice of subvarieties of regular double p-algebras.

9.1 H^+-algebra versus double Heyting algebras

In Example 3.1.6, we gave a lattice $\mathbf{L}$ that has an H^+-subuniverse that is not a double Heyting subuniverse. That subuniverse has seven elements, and it is easily verified that there are no double Heyting subuniverses of $\mathbf{L}$ that have seven elements. So it is more accurate to say that the set of isomorphism classes of the lattice reducts of double Heyting subalgebras of $\mathbf{L}$ is different to the set of isomorphism classes of the lattice reducts of H^+ subalgebras of $\mathbf{L}$.

Open Problem 16. Characterise the finite distributive lattices for which the double Heyting subuniverses and H^+-subuniverses differ. We leave the definition of "differ" open to interpretation.

It follows from Jónsson's Lemma that the lattice reducts of algebras in the variety generated by the lattice $\mathbf{L}$ above will differ depending on whether it is treated as a double Heyting algebra or an H^+-algebra. This suggests, but it does not imply, that the lattice of subvarieties of $\mathcal{H}^+$ and the lattice of subvarieties of $\mathcal{DH}$ are non-isomorphic.

Open Problem 17. What relationship exists between $\mathcal{L}(\mathcal{H}^+)$ and $\mathcal{L}(\mathcal{DH})$? Does one embed into the other, or are they possibly isomorphic?

Given that every double Heyting algebra defines an H$^+$-algebra, we can ask a sort of converse question: for every H$^+$-algebra $\mathbf{A}$, is there a double Heyting algebra $\mathbf{B}$ such that $\mathbf{A}$ embeds into $\mathbf{B}^\flat$? The answer to that question is yes. Let $\mathbf{A}$ be an H$^+$-algebra. The canonical extension of $\mathbf{A}$ is given by the lattice of upsets (rather than clopen upsets) of $\mathcal{F}_p(\mathbf{A})$, and it is easy to see that it forms a double Heyting algebra that $\mathbf{A}$ embeds into. That being said, the canonical extension of $\mathbf{A}$ can have more congruences. Thus, we sharpen the question as follows.

Open Problem 18. Let $\mathbf{A}$ be an H$^+$-algebra that does not form a double Heyting algebra. Does there exist a double Heyting algebra $\mathbf{B}$ such that $\mathbf{Con}(\mathbf{A}) \cong \mathbf{Con}(\mathbf{B})$? Moreover, can we construct $\mathbf{B}$ such that $\mathbf{Con}(\mathbf{A}) \cong \mathbf{Con}(\mathbf{B})$ *and* $\mathbf{A}$ embeds into $\mathbf{B}^\flat$?

A possible answer may be given by Dedekind–MacNeille completions. It is known that the Dedekind–MacNeille completion of a Heyting algebra also forms a Heyting algebra (see [41]). It follows that the Dedekind–MacNeille completion of a double Heyting algebra is also a double Heyting algebra. It is not clear whether this applies to H$^+$-algebras as well.

Open Problem 19. Does the Dedekind–MacNeille completion of an H$^+$-algebra form an H$^+$-algebra? Does it happen to form a double Heyting algebra as well?

Moreover, Harding and Bezhanishvili [41] proved that the only varieties of Heyting algebras closed under Dedekind–MacNeille completions are the trivial subvariety, the variety of Boolean algebras, and the variety of all Heyting algebras. On the other hand, they claim without proof that their methods can be used to show that the variety of double Heyting algebras generated by the three-element chain is closed under Dedekind–MacNeille completions. We have not been able to reconstruct a proof using their methods.

Open Problem 20. Characterise the subvarieties of $\mathcal{H}^+$ and $\mathcal{DH}$ that are closed under Dedekind–MacNeille completions.

9.2 The graph reconstruction conjecture

Definition 9.2.1. A *simple graph* is a graph without loops. Let G be a finite simple graph. A *vertex-deleted subgraph* of G is a subgraph of G obtained by deleting a single vertex and all of its incident edges. The *deck* of G, denoted by $D(G)$, is the set of vertex-deleted subgraphs of G. A finite simple graph H is a *reconstruction* of G if there exists a bijection $f\colon D(G) \to D(H)$ such that $K \cong f(K)$, for all $K \in D(G)$.

See Figure 9.1 for some examples of some graphs and their decks. Informally, a graph is a reconstruction of another graph if the two graphs have the same deck.

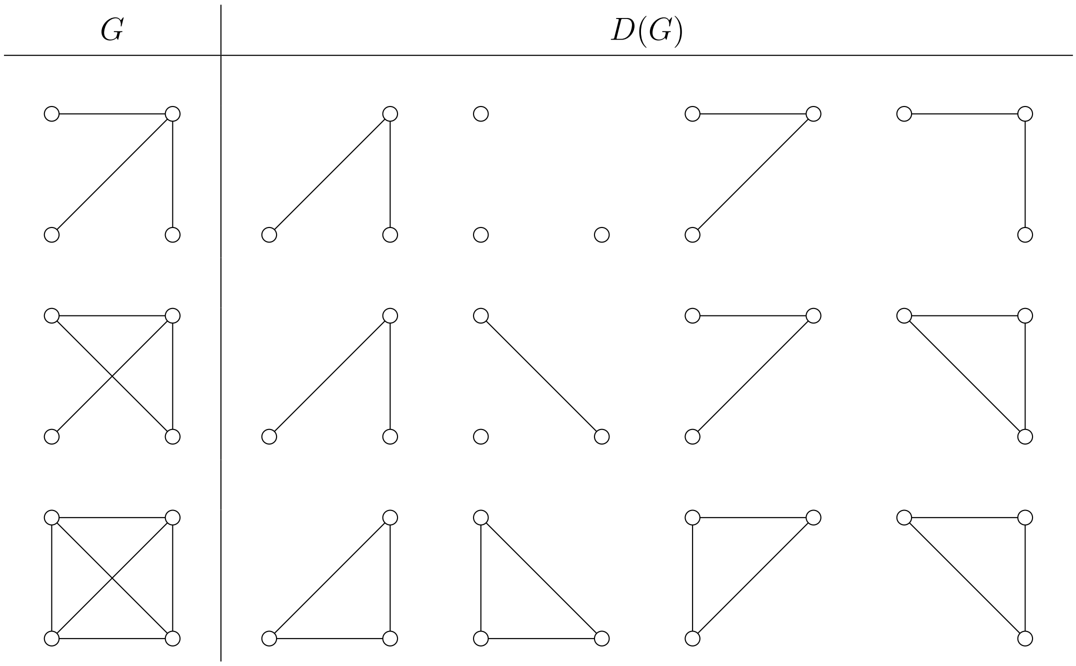


Figure 9.1: Some graphs and their decks. Observe that in each case, the deck contains a number of isomorphic subgraphs.

Graph Reconstruction Problem. If G is a finite simple graph and H is a reconstruction of G, is H isomorphic to G?

The *graph reconstruction conjecture*, first posed by Kelly [56] and Ulam [87], postulates that the answer is yes, provided that the graphs have more than two vertices—both graphs with two vertices have the same deck. For a survey of the graph reconstruction conjecture, refer to Harary [40] or Bondy [15].

Definition 9.2.2. A finite simple graph G is *reconstructible* if every reconstruction of G is isomorphic to G.

It has been verified that all (simple) graphs with at most 11 vertices are reconstructible [68]. Bollobás [14] proved that almost all graphs are uniquely determined by three elements of their deck, and it follows that almost all graphs are reconstructible. Some classes of graphs have been proven reconstructible as well.

For instance, it is easy to show that degree-regular graphs are reconstructible (see also [40]). The list of reconstructible classes also includes disconnected graphs [40], trees [56], unit interval graphs [90], nearly acyclic graphs [67], and outerplanar graphs [37]. On the other hand, counterexamples are known for more general versions of the graph reconstruction conjecture. For example, directed graphs are not always reconstructible [82], nor are hypergraphs [57], or infinite graphs [32, 33].

We will see now that the problem can be translated into a question about lattices. Recall that the lattice of subgraphs of an incidence structure G is denoted by $\mathcal{S}(G)$. Two incidence structures $\langle P_1, L_1, I_1\rangle$ and $\langle P_2, L_2, I_2\rangle$ are *isomorphic* if there exists a pair of bijective maps $\varphi_1\colon P_1 \to P_2$ and $\varphi_2\colon L_1 \to L_2$ such that $(x, y) \in I_1$ if and only if $(\varphi_1(x), \varphi_2(y)) \in I_2$. The next two results are obvious.

Proposition 9.2.3. *Let G be an incidence structure and let $H \in \mathcal{S}(G)$. The ordered sets ${\downarrow}H$ and $\mathcal{S}(H)$ are equal.*

Proposition 9.2.4. *Let G and H be incidence structures with no empty lines. Then $\mathcal{S}(H)$ and $\mathcal{S}(G)$ are isomorphic if and only if G and H are isomorphic.*

Let G be a finite simple graph and let v be a vertex in G. Observe that the subgraph obtained by deleting v is exactly the pseudocomplement of $\langle\{v\}, \varnothing\rangle$ in $\mathcal{S}(G)$. Moreover, the subgraph $\langle\{v\}, \varnothing\rangle$ is an atom in $\mathcal{S}(G)$, and every atom in $\mathcal{S}(G)$ is of this form. Thus, knowing the deck of a graph G is equivalent to knowing the downsets of pseudocomplements of atoms in $\mathcal{S}(G)$. We can make this more precise.

Proposition 9.2.5. *Let G be a finite incidence structure and assume G has no empty lines. For every point p of G, the set ${\downarrow}\neg\langle\{p\}, \varnothing\rangle$ is a minimal prime ideal of $\mathcal{S}(G)$. Moreover, every minimal prime ideal of $\mathcal{S}(G)$ is of this form.*

Proof. Since G is finite and $\mathcal{S}(G)$ is distributive, prime ideals in $\mathcal{S}(G)$ are exactly the principal ideals generated by a meet-irreducible element. So we will show that the minimal meet-irreducible elements in $\mathcal{S}(G)$ are exactly the substructures $\neg\langle\{p\}, \varnothing\rangle$, for some point p. For simplicity, we will identify a point p with the substructure $\langle\{p\}, \varnothing\rangle$. Let p be a point in G. Because p is an atom, $\neg p$ is meet-irreducible and so ${\downarrow}\neg p$ is a prime ideal. To see that it is minimal, we will show that every element below $\neg p$ is not meet-irreducible. If $K = \langle P_K, L_K\rangle \le \neg p$, then p is not a point in K. Then $K_v := \langle P_K \cup \{p\}, L_K\rangle \neq K$, and $K_p \wedge \neg p = K$, so K is not meet-irreducible.

For the last claim, let K be a meet-irreducible element of $\mathcal{S}(G)$. If K is missing a point p, then $K \wedge p = 0$, so $K \le \neg p$. Then by the minimality of $\neg p$, we have $K = \neg p$. Otherwise, because G has no empty lines, K contains every point of G

and is missing at least one line. Because K is meet-irreducible, it is missing exactly one line. Let p be a point incident to the missing line. Then $\neg p \leq K$, so K is a not minimal join-irreducible. □

Definition 9.2.6. Let $\mathbf{L}$ be a lattice. Define the *deck* of $\mathbf{L}$, denoted by $D(\mathbf{L})$, to be the set of minimal prime ideals of $\mathbf{L}$. We say that a lattice $\mathbf{K}$ is a *reconstruction* of $\mathbf{L}$ if there is a bijection $f\colon D(\mathbf{L}) \to D(\mathbf{K})$ such that I and $f(I)$ are order-isomorphic, for every $I \in D(\mathbf{L})$.

Then the graph reconstruction problem is equivalent to the following lattice-theoretic problem.

Graph Reconstruction Problem for Lattices. Let $\mathcal{K}$ be the class of all lattices $\mathbf{L}$ such that

(1) $\mathbf{L}$ is a finite distributive lattice,

(2) there is no chain of three or more join-irreducible elements in $\mathbf{L}$,

(3) every join-irreducible element in $\mathbf{L}$ is either an atom or has exactly two atoms below it.

If two lattices in $\mathcal{K}$ are reconstructions of each other, are they isomorphic?

Note that the first two conditions imply that $\mathbf{L}$ is the underlying lattice of a finite regular double p-algebra, and the third guarantees that it is isomorphic to the lattice of subgraphs of a simple graph. We have made no further steps in resolving the conjecture. We find it plausible that it could be cracked using lattice-theoretic tools, since the conditions imposed above are rather strong, especially compared to the complete lack of conditions imposed on graphs.

9.3 Small subvarieties

We saw in Chapter 8 that the smallest subvarieties of both $\mathcal{H}^+$ and $\mathcal{DH}$ are, in increasing order, the trivial subvariety, Var($\mathbf{2}$), and Var($\mathbf{3}$). What happens at the next level is completely unclear. For example, whether or not every cover of Var($\mathbf{3}$) must be finitely generated is not known.

Open Problem 21. If $\mathcal{V}$ is a variety of H$^+$-algebras or double Heyting algebras and $\mathcal{V}$ has finitely many subvarieties, is $\mathcal{V}$ finitely generated? It is worth noting that the answer to this question for Heyting algebras is *yes*, as proved by Day [27].

In the meantime, we will focus on finitely generated subvarieties. Let $\mathbf{A}$ be a finite subdirectly irreducible double Heyting algebra. Then $\mathbf{A}$ is simple by Corollary 1.5.8. By Lemma 8.2.1 and Jónsson's Lemma, if $\mathbf{B}$ is a subdirectly irreducible double Heyting algebra, then $\mathbf{B} \in \mathrm{Var}(\mathbf{A})$ if and only if $\mathbf{B} \leq \mathbf{A}$. So, $\mathrm{Var}(\mathbf{A})$ covers $\mathrm{Var}(\mathbf{3})$ if and only if $\mathsf{IS}(\mathbf{A}) = \mathsf{I}(\{\mathbf{2}, \mathbf{3}, \mathbf{A}\})$. Using that condition, we ran a computer search to find examples of small ordered sets X such that $\mathrm{Var}(\mathcal{U}(X))$ covers $\mathrm{Var}(\mathbf{3})$. Figure 9.2 shows the results. While each of the corresponding algebras have only three double Heyting subuniverses (up to isomorphism), at least one of the algebras found has a fourth H^+-subuniverse.

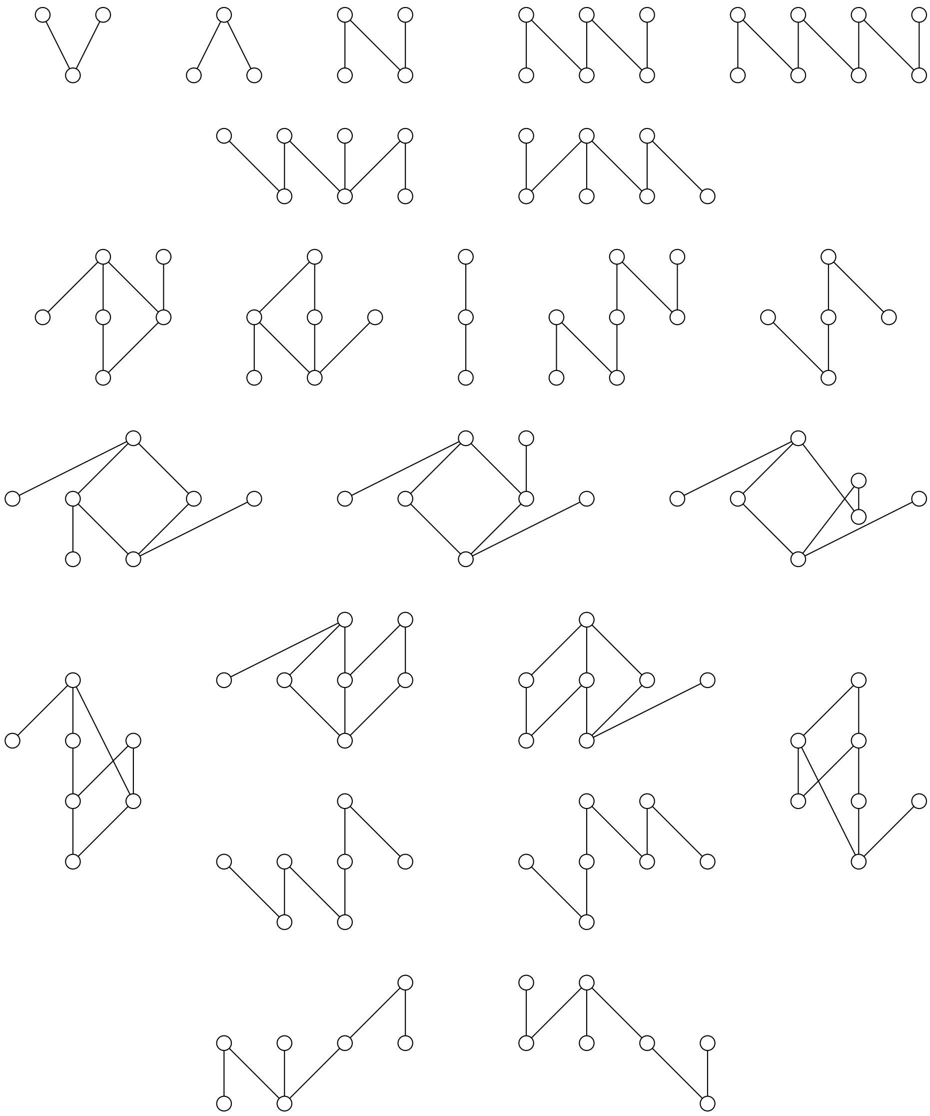


Figure 9.2: For each of the ordered sets above, the lattice of upsets is a double Heyting algebra with (up to isomorphism) exactly three subalgebras.

One may be led to expect from Figure 9.2 that every fence with one down-tail and one up-tail has no non-trivial subalgebras. This is not the case, because there is a surjective H^+-morphism from the ten-element fence to the four-element fence. This brings us to the lattice of subvarieties of regular double p-algebras, because the lattice of upsets of a fence forms a regular double p-algebra. Subalgebras of finite regular double p-algebras may be accessible by a geometric approach using the results of Chapter 4. Let $G = \langle P, L, I\rangle$ be an incidence structure and assume P and L are disjoint. Let $X = P \cup L$ and define an order $\leq$ on X by $x \leq y$ if and only if $(y, x) \in I$ or $x = y$. It is easy to see that every element of X is minimal or maximal. We will call X the *incidence order of* G. Recall from Section 4.3 that $\mathcal{E}(X)$ is the incidence structure $\langle V, E, J\rangle$ defined by

$$\begin{aligned} V &= \max(X), \\ E &= X \backslash \max(X), \\ J &= \{(x, y) \in V \times E \mid x > y\}. \end{aligned}$$

Clearly, $\mathcal{E}(X) = G$. By Theorem 4.3.2, if Y is an ordered set such that every element is minimal or maximal, then $\mathcal{S}(\mathcal{E}(Y)) \cong \mathcal{U}(Y)$. Hence, we obtain the next result.

Proposition 9.3.1. *Let G be a finite incidence structure. Then $\mathcal{F}_p(\mathcal{S}(G))$ and the incidence order of G are isomorphic.*

Definition 9.3.2. Let G be a finite incidence structure. We say that G is *connected* if the incidence order of G is connected.

Proposition 8.1.2 and Corollary 1.5.8 prove the next result.

Proposition 9.3.3. *Let G be a finite incidence structure. The following are equivalent:*

(1) *G is connected;*

(2) *$\mathcal{S}(G)$ is simple;*

(3) *$\mathcal{S}(G)$ is subdirectly irreducible.*

Thus, to discover covers of $\mathrm{Var}(\mathbf{3})$ in $\mathcal{RDP}$, we turn our attention to finite connected incidence structures. Let G be a finite incidence structure. By Theorem 1.6.4, double p-subalgebras, H^+-subalgebras, and double Heyting subalgebras of $\mathcal{S}(G)$ are all the same. So to obtain subalgebras of $\mathcal{S}(G)$, we consider double Heyting morphisms. A straightforward translation of Definition 2.2.6 applied to the incidence order yields the next result.

Proposition 9.3.4. *Let $G = \langle P, L, I\rangle$ be a finite connected incidence structure and assume P and L are disjoint. Every subalgebra of $\mathcal{S}(G)$ corresponds to either the two-element Boolean algebra or a structure $\langle X, Y, J\rangle$ and a surjective map $\varphi\colon P \cup L \to X \cup Y$ such that $X \cap Y = \varnothing$ and, for all $x \in P$ and all $y \in L$,*

(1) *if $(x, y) \in I$, then $(\varphi(x), \varphi(y)) \in J$,*

(2) *if $(\varphi(x), \varphi(y)) \in J$, then there exists $p \in P$ and $l \in L$ such that*

   (i) *$(x, l) \in I$ and $(p, y) \in I$,*

   (ii) *$\varphi(p) = \varphi(x)$ and $\varphi(l) = \varphi(y)$.*

Definition 9.3.5. Let $\mathbf{A}$ be a finite simple regular double p-algebra. We say that a subalgebra of $\mathbf{A}$ is *obvious* if it is isomorphic to any of $\mathbf{2}$, $\mathbf{3}$, or $\mathbf{A}$.

Let $G = \langle P, L, I\rangle$ be an incidence structure. It is easy to see that every isomorphism from G to G satisfies the conditions of Proposition 9.3.4. So, assuming that $\mathcal{S}(G)$ is non-trivial and not isomorphic to either $\mathbf{2}$ or $\mathbf{3}$, if there exists a non-trivial automorphism on G, then $\mathcal{S}(G)$ has a non-obvious subalgebra. Simple graphs that have no non-trivial automorphisms are known as *asymmetric graphs*. We have checked some examples of asymmetric graphs by hand to see if their subgraph lattices have any non-obvious subalgebras. They all did.

Definition 9.3.6. Let us say that a connected incidence structure G is *strongly asymmetric* if $\mathcal{S}(G)$ has no non-obvious subalgebras.

The next problem is equivalent to finding all finitely generated covers of $\mathrm{Var}(\mathbf{3})$ in $\mathcal{L}(\mathcal{RDP})$.

Open Problem 22. Characterise strongly asymmetric incidence structures. We conjecture that no simple graphs are strongly asymmetric.

List of open problems

Open Problem 1 (page 25). Is the variety of regular double p-algebras generated by its finite members? The proof of Theorem 3.1.4 will not apply to regular double p-algebras. This is because, by Theorem 1.6.2, the underlying lattice of a finite regular double p-algebra must have no chain of three or more join-irreducible elements, and that is not a property preserved by sublattices.

Open Problem 2 (page 26). Let $n \in \omega$, let $\mathbf{A}$ be an H^+-algebra, and assume every chain of prime filters of $\mathbf{A}$ has at most n elements. Does the underlying lattice of $\mathbf{A}$ form a double Heyting algebra? More generally, what conditions can ensure a Heyting algebra or an H^+-algebra also forms a double Heyting algebra?

Open Problem 3 (page 30). Find an algebraic description of varieties whose subvariety lattices form a Heyting algebra.

Open Problem 4 (page 33). Let $\langle X; \mathcal{T} \rangle$ be a topological space and assume that X is T_0 but not T_1. When does $\mathcal{T}$ form a double Heyting algebra? More generally, when is $\mathcal{T}$ dually pseudocomplemented? Is it a topologically meaningful assumption? It would also be interesting to find a topological space that is dually pseudocomplemented but does not form a double Heyting algebra.

Open Problem 5 (page 39). What is the equational theory of the double Heyting algebras $\mathcal{C}$ and $\mathcal{C}_S$? Interestingly, the two lattices are both simple double Heyting algebras. This is because the bottom of $\mathcal{C}$ is completely meet-irreducible, so $\neg{\sim}[G]_{\mathcal{C}} = 0$, for all $[G]_{\mathcal{C}} \neq 1$. With that in mind, give a subdirect product representation of the double Heyting algebras obtained by truncating $\mathcal{C}$ and $\mathcal{C}_S$ at the bottom.

Open Problem 6 (page 39). The direct product of digraphs defined in this thesis is also known as the *categorical product.* True to the name, it is the categorical product with respect to digraph homomorphisms. Various other products are studied in graph theory. There is an enormous literature on these products, described extensively by Hammack, Imrich and Klavžar [39]. The so-called "four standard graph products" are the direct product, the Cartesian product, the strong product, and the lexicographic product (see [39]). The exponential graph is the Heyting implication, or in other words, it is residual of the direct product in $\mathcal{C}$. It has been rumoured that the Cartesian product is also residuated in $\mathcal{C}$. We have not yet investigated this, and we have heard no rumours on whether the strong product and lexicographic product are also residuated.

Open Problem 7 (page 45). In terms of the incidence structure, how can we describe subalgebras and homomorphic images of the double p-algebra $\mathcal{S}(G)$?

Open Problem 8 (page 78). Can Corollary 6.1.17 be generalised with $\sim$ in the signature but with $\dot{-}$ excluded?

Open Problem 9 (page 85). Generalise Theorem 6.3.1 so that it incorporates the results for the residuated lattices considered by Kowalski [60], Takamura [83], and Kowalski and Ferreirim [61].

Open Problem 10 (page 91). Every semisimple variety of monadic Heyting algebras is a discriminator variety. But they are not dually pseudocomplemented, so this fact does not follow from Theorem 6.3.1. With that in mind, find a common generalisation, perhaps also including the residuated lattices mentioned in Open Problem 9.

Open Problem 11 (page 95). Is $\mathcal{SHRA}$ a discriminator variety? One way to prove otherwise is to observe that, because of the double Heyting algebra reduct, finite subdirectly irreducible SHRAs are simple. Then, to show that $\mathcal{SHRA}$ is not a discriminator variety, it would be sufficient to exhibit, for each $n \in \omega$, a finite SHRA—perhaps based on a fence—that does not satisfy $d^n x = 0$.

Open Problem 12 (page 101). Are there infinitely many covers of Var($\mathbf{3}$) in $\mathcal{L}(\mathcal{DH})$? Characterise them. What about $\mathcal{H}^+$ and $\mathcal{RDP}$? A different approach using algebras of incidence structures and fences may work for $\mathcal{RDP}$.

Open Problem 13 (page 109). Are there any infinite splitting algebras in $\mathcal{RDP}$?

Open Problem 14 (page 111). Are the forbidden tree varieties generated by their finite members? This generalises the same problem for regular double p-algebras, since $\mathcal{RDP}$ is term-equivalent to both Forb($\mathcal{H}^+, 3$) and Forb($\mathcal{DH}, 3$).

Open Problem 15 (page 112). If T is a tree with a maximal element covering the root, then Forb($\mathcal{DH}, T$) and Forb($\mathcal{H}^+, T$) are varieties, but Theorem 8.2.12 does not apply. Does this suggest there are non-trivial splitting algebras in those classes, or is a different construction required?

Open Problem 16 (page 113). Characterise the finite distributive lattices for which the double Heyting subuniverses and H^+-subuniverses differ. We leave the definition of "differ" open to interpretation.

Open Problem 17 (page 114). What relationship exists between $\mathcal{L}(\mathcal{H}^+)$ and $\mathcal{L}(\mathcal{DH})$? Does one embed into the other, or are they possibly isomorphic?

Open Problem 18 (page 114). Let $\mathbf{A}$ be an H^+-algebra that does not form a double Heyting algebra. Does there exist a double Heyting algebra $\mathbf{B}$ such that $\mathbf{Con}(\mathbf{A}) \cong \mathbf{Con}(\mathbf{B})$? Moreover, can we construct $\mathbf{B}$ such that $\mathbf{Con}(\mathbf{A}) \cong \mathbf{Con}(\mathbf{B})$ *and* $\mathbf{A}$ embeds into $\mathbf{B}^\flat$?

Open Problem 19 (page 114). Does the Dedekind–MacNeille completion of an H^+-algebra form an H^+-algebra? Does it happen to form a double Heyting algebra as well?

Open Problem 20 (page 114). Characterise the subvarieties of $\mathcal{H}^+$ and $\mathcal{DH}$ that are closed under Dedekind–MacNeille completions.

Open Problem 21 (page 117). If $\mathcal{V}$ is a variety of H^+-algebras or double Heyting algebras and $\mathcal{V}$ has finitely many subvarieties, is $\mathcal{V}$ finitely generated? It is worth noting that the answer to this question for Heyting algebras is *yes*, as proved by Day [27].

Open Problem 22 (page 120). Characterise strongly asymmetric incidence structures. We conjecture that no simple graphs are strongly asymmetric.

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