Filters and ideals in double-Heyting algebras

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Definition

Let **A** be a bounded lattice. It is a *relatively pseudocomplemented lattice* if, for every $x, y \in A$, there exists an element $x \to y$ satisfying

$$x \wedge z \leq y \iff z \leq x \to y.$$

It is a *dual relatively pseudocomplemented lattice* if, for every $x, y \in A$, there exists an element $y \dot{-} x$ satisfying

$$x \vee z \geq y \iff z \geq y \dot{-} x,$$

and it is a *dually pseudocomplemented lattice* if, for every $x \in A$, there exists an element $\sim x$ such that

$$x \vee y = 1 \iff y \geq \sim x.$$

Definition

A *Heyting algebra* has the fundamental operations $\{\vee, \wedge, \rightarrow, 0, 1\}$. Let $x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$. For any lattice $\mathbf{A}$, identify the set $\mathrm{Fil}(\mathbf{A})$ of filters of $\mathbf{A}$ with the ordered set $\langle \mathrm{Fil}(\mathbf{A}), \subseteq \rangle$.

Actually, $\mathrm{Fil}(\mathbf{A})$ is a lattice.

Theorem

Let $\mathbf{A}$ *be a Heyting algebra. Then* $\mathrm{Con}(\mathbf{A}) \cong \mathrm{Fil}(\mathbf{A})$*, via the maps*

$$\theta \colon F \mapsto \{(x, y) \mid x \leftrightarrow y \in F\},$$
$$\theta^{-1} \colon \alpha \mapsto 1/\alpha = \{x \in A \mid x \equiv_\alpha 1\}.$$

Definition

A *dually pseudocomplemented Heyting algebra* (H^+ algebra for short) is an algebra **A** with fundamental operations $\{\vee, \wedge, \rightarrow, \sim, 0, 1\}$. A *normal filter* is a filter $F \subseteq A$ satisfying

$$x \in F \implies dx := \neg{\sim}x \in F,$$

where $\neg x = x \rightarrow 0$. Let $\mathrm{NF}(\mathbf{A})$ denote the ordered set of normal filters.

Theorem (Sankappanavar, 1985)

Let **A** *be a* H^+ *algebra. Then* $\mathrm{Con}(\mathbf{A}) \cong \mathrm{NF}(\mathbf{A})$*, via the maps*

$$\theta \colon F \mapsto \{(x, y) \mid x \leftrightarrow y \in F\},$$
$$\theta^{-1} \colon \alpha \mapsto 1/\alpha = \{x \in A \mid x \equiv_\alpha 1\}.$$

Definition

A *double Heyting algebra* is an algebra $\mathbf{A}$ with fundamental operations $\{\vee, \wedge, \rightarrow, \dot{-}, 0, 1\}$. It defines a H^+ algebra by $\sim x = 1 \dot{-} x$, so let $\mathbf{A}^\flat$ denote the H^+ term reduct.

Theorem (Sankappanavar, 1985)

Let $\mathbf{A}$ *be a double Heyting algebra. Then* $\mathrm{Con}(\mathbf{A}) = \mathrm{Con}(\mathbf{A}^\flat)$.

Corollary (Köhler, 1980)

Let $\mathbf{A}$ *be a double Heyting algebra. Then* $\mathrm{Con}(\mathbf{A}) \cong \mathrm{NF}(\mathbf{A})$*, via*

$$\theta\colon F \mapsto \{(x, y) \mid x \leftrightarrow y \in F\},$$
$$\theta^{-1}\colon \alpha \mapsto 1/\alpha = \{x \in A \mid x \equiv_\alpha 1\}.$$

Definition

Let **A** be a H^+ algebra and let I be an ideal of **A**. Then I is a *normal ideal* if $x \in I$ implies $qx := \sim\neg x \in I$. Let $\mathrm{NI}(\mathbf{A})$ denote the ordered set of normal ideals of **A**. Define $x \div y = (x \dot{-} y) \vee (y \dot{-} x)$.

Corollary

Let **A** *be a double Heyting algebra. Then* $\mathrm{Con}(\mathbf{A}) \cong \mathrm{NI}(\mathbf{A})$*, via*

$$\lambda \colon I \mapsto \{(x, y) \mid x \div y \in I\},$$
$$\lambda^{-1} \colon \alpha \mapsto 0/\alpha = \{x \in A \mid x \equiv_\alpha 0\}.$$

Hence,

$$\mathrm{NF}(\mathbf{A}) \underset{\overleftarrow{\theta^{-1}}}{\overset{\overrightarrow{\theta}}{\cong}} \mathrm{Con}(\mathbf{A}) \underset{\overleftarrow{\lambda}}{\overset{\overrightarrow{\lambda^{-1}}}{\cong}} \mathrm{NI}(\mathbf{A}).$$

Question: can we say more about $\theta^{-1} \circ \lambda$ and $\lambda^{-1} \circ \theta$?

Forget about congruences for now.

Definition

Let **A** be a H^+ algebra and for each filter F and ideal I define $\mathcal{I}(F)$ and $\mathcal{F}(I)$ by

$$\mathcal{I}(F) = {\downarrow}{\sim}F = \{y \in A \mid (\exists x \in F)\ y \leq {\sim}x\}, \text{ and,}$$
$$\mathcal{F}(I) = {\uparrow}\neg I = \{y \in A \mid (\exists x \in I)\ y \geq \neg x\}.$$

Lemma

*Let **A** be a H^+ algebra, let I be an ideal and let F be a filter of **A**.*

1. *$\mathcal{I}(F)$ is an ideal and $\mathcal{F}(I)$ is a filter.*
2. *If F is normal then $\mathcal{I}(F)$ is normal.*
3. *If I is normal then $\mathcal{F}(I)$ is normal.*

Theorem

Let **A** *be a* H^+ *algebra. Then* NF(**A**) *and* NI(**A**) *are order-isomorphic via the mutually inverse maps* $\mathcal{I}$ *and* $\mathcal{F}$.

Corollary

Let **A** *be a* H^+ *algebra. Then*

$$\mathrm{Con}(\mathbf{A}) \underset{\overleftarrow{\theta}}{\overset{\overrightarrow{\theta^{-1}}}{\cong}} \mathrm{NF}(\mathbf{A}) \underset{\overleftarrow{\mathcal{F}}}{\overset{\overrightarrow{\mathcal{I}}}{\cong}} \mathrm{NI}(\mathbf{A}).$$

Question: what are the maps $\theta \circ \mathcal{F}$ and $\mathcal{I} \circ \theta^{-1}$? The asymmetry of H^+ algebras means that λ isn't defined in this setting.

Back to first principles:

Definition

Let **A** be a lattice, let $F \subseteq A$ be a filter and let $I \subseteq A$ be an ideal. Define $\theta_L(F)$ and $\lambda_L(I)$ by

$$\theta_L(F) = \{(x, y) \in A^2 \mid (\exists f \in F)\ x \wedge f = y \wedge f\},$$
$$\lambda_L(I) = \{(x, y) \in A^2 \mid (\exists i \in I)\ x \vee i = y \vee i\}.$$

Fact

Let **A** be a lattice. The following are equivalent.

1. **A** is distributive.
2. For every filter $F \subseteq A$, the relation $\theta_L(F)$ is a lattice congruence.
3. For every ideal $I \subseteq A$, the relation $\lambda_L(I)$ is a lattice congruence.

Let **A** be a (double) Heyting algebra. Recall that

$$\theta(F) = \{(x, y) \mid x \leftrightarrow y \in F\},$$
$$\lambda(I) = \{(x, y) \mid x \div y \in F\}.$$

Lemma

Let **A** *be a (double) Heyting algebra. Then,*

1. $x \leftrightarrow y = \max\{z \in A \mid x \wedge z = y \wedge z\}$,
2. $x \div y = \min\{z \in A \mid x \vee z = y \vee z\}$.

Corollary

If **A** *is a (double) Heyting algebra then* $\theta(F) = \theta_L(F)$ *and* $\lambda(I) = \lambda_L(I)$.

Theorem

Let **A** *be a* H^+ *algebra and let* I *be a normal filter of* **A**. *Then,*

$$\lambda_L(I) = \{(x, y) \mid \sim(x \leftrightarrow y) \in I\}.$$

Moreover, $\theta \circ \mathcal{F}(I) = \lambda_L(I)$.

Corollary

Let **A** *be a double Heyting algebra and let* F *be a normal filter of* **A**. *The following are equivalent.*

1. $x \leftrightarrow y \in F$,
2. $\sim(x \leftrightarrow y) \in \mathcal{I}(F)$,
3. $x \div y \in \mathcal{I}(F)$,
4. $\neg(x \div y) \in F$.

Let's expand the signature further.

Theorem

Let $\mathbf{A} = \langle A; \vee, \wedge, \to, f, 0, 1\rangle$ *and assume* f *satisfies*

$$f(1) = 1, \qquad f(x \wedge y) = f(x) \wedge f(y).$$

Let F *be a filter of* $\mathbf{A}$*. Then* $\theta(F)$ *is a congruence if and only if* F *is closed under* f*.*

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The proof relies on the following fact.

Lemma

With f *as above,* $f(x \rightarrow y) \leq f(x) \rightarrow f(y)$*.*

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Let F *be a filter of* $\mathbf{A}$. *Then* $\theta(F)$ *is a congruence if and only if* F *is closed under* f.

The proof relies on the following fact.

Lemma

With f *as above,* $f(x \rightarrow y) \leq f(x) \rightarrow f(y)$.

- The proof does not work for join-preserving operations.
- The proof does generalise to *n*-ary maps (use whiteboard).

Definition

An algebra $\mathbf{B} = \langle B, \{f_1, \dots, f_n\}, \vee, \wedge, \neg, 0, 1\rangle$ is a *boolean algebra with operators* (BAO for short) if the obvious operations form a boolean algebra, and for each $i \leq n$, the operation f_i satisfies

$$f_i(\dots, 0, \dots) = 0,$$
$$f_i(\dots, x \vee y, \dots) = f(\dots, x, \dots) \vee f(\dots, y, \dots),$$

for each coordinate.

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for each coordinate.

Corollary

If $\mathbf{B}$ *is a BAO and (for convenience) each* f_i *is unary, then* $\lambda(I)$ *is a congruence if and only if*

$$x \in I \implies f_1 x \vee f_2 x \vee \ldots \vee f_n x \in I.$$

Example

An algebra $\mathbf{B} = \langle B, \vee, \wedge, \circ, \neg, \smile, 0, 1, \mathrm{id}\rangle$ is a *relation algebra* if it is a BAO further satisfying

1. $\langle A, \circ, \mathrm{id}\rangle$ is a monoid,
2. $\smile\smile x = x$,
3. $\smile(x \circ y) = \smile y \circ \smile x$, and,
4. $(x \circ y) \wedge z = 0 \iff (\smile x \circ z) \wedge y = 0 \iff (z \circ \smile y) \wedge x = 0$.

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Corollary

$\lambda(I)$ is a congruence if and only if I is closed under $\smile x \vee (1 \circ x) \vee (x \circ 1)$.

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$\lambda(I)$ *is a congruence if and only if* I *is closed under* $\smile x \vee (1 \circ x) \vee (x \circ 1)$.

Fun fact

The variety of relation algebras is a discriminator variety.

Lemma

Let **A** *be a* H^+ *algebra, let* F *be a normal filter of* **A**, *let* I *be a normal ideal, and let* t *be an order preserving unary map.*

1. F *is closed under* f *if and only if* $\mathcal{I}(F)$ *is closed under* $\sim f \neg$.
2. I *is closed under* f *if and only if* $\mathcal{F}(I)$ *is closed under* $\neg f \sim$.

Lemma

*Let **A** be a H^+ algebra, let F be a normal filter of **A**, let I be a normal ideal, and let t be an order preserving unary map.*

1. *F is closed under f if and only if $\mathcal{I}(F)$ is closed under $\sim f\neg$.*
2. *I is closed under f if and only if $\mathcal{F}(I)$ is closed under $\neg f\sim$.*

Treat the previous stuff more generally: let **A** be an algebra with a Heyting algebra term reduct and assume there is another term *t* in the language of **A** that determines congruences

Definition

Let **A** be an algebra with a Heyting algebra term reduct. A filter $F \subseteq A$ is called a *congruence filter* if $\theta(F)$ is a congruence of **A**.

Definition

Let $\mathbf{A}$ be an algebra with a Heyting algebra term reduct. A filter $F \subseteq A$ is called a *congruence filter* if $\theta(F)$ is a congruence of $\mathbf{A}$. A term t in the language of $\mathbf{A}$ is called a *congruence filter term* if:

1. $t^{\mathbf{A}}$ is order preserving, and
2. $\theta(F)$ is a congruence if and only if F is closed under $t^{\mathbf{A}}$.

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If $\sim$ is present then using the previous theorem we can flip between congruence filters and “congruence ideals”.

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If $\sim$ is present then using the previous theorem we can flip between congruence filters and "congruence ideals". But we need $\dot{-}$ to generalise BAOs!

Corollary

Let $\mathbf{A} = \langle A; \vee, \wedge, \to, \dot{-}, f, 0, 1 \rangle$ *and assume* f *satisfies*

$$f(0) = 0, \qquad f(x \vee y) = f(x) \vee f(y).$$

Let F *be a filter of* $\mathbf{A}$. *Then* $\theta(F)$ *is a congruence if and only if* F *is closed under* t, *where* $tx = \neg{\sim}x \wedge \neg f{\sim}x$.

Corollary

Let $\mathbf{A} = \langle A; \vee, \wedge, \rightarrow, \dot{-}, f, 0, 1\rangle$ *and assume f satisfies*

$$f(0) = 0, \qquad f(x \vee y) = f(x) \vee f(y).$$

Let F be a filter of $\mathbf{A}$*. Then* $\theta(F)$ *is a congruence if and only if F is closed under t, where* $tx = \neg{\sim}x \wedge \neg f{\sim}x$*.*

Open problem

Replace $\mathbf{A}$ with $\langle A; \vee, \wedge, \rightarrow, \sim, f, 0, 1\rangle$. Does the same term work?

An application

A *symmetric Heyting algebra* is an algebra $\langle A; \vee, \wedge, \to, \frown, 0, 1\rangle$ such that $\frown$ is a dual automorphism on **A**. This defines a double Heyting algebra by $y \dot{-} x = \frown(\frown x \to \frown y)$.

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Fact

$\theta(F)$ is a congruence if and only if $x \in F$ implies $\neg\frown x \in F$.

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A *symmetric Heyting algebra* is an algebra $\langle A; \vee, \wedge, \to, \frown, 0, 1\rangle$ such that $\frown$ is a dual automorphism on **A**. This defines a double Heyting algebra by $y \dot{-} x = \frown(\frown x \to \frown y)$.

Fact

$\theta(F)$ is a congruence if and only if $x \in F$ implies $\neg\frown x \in F$.

A *symmetric Heyting relation algebra* (SHRA) extends that signature with a unary operation id and a binary operation $\circ$ satisfying

1. $(A, \circ, \mathrm{id})$ is a monoid,
2. $\sim\frown(x \circ y) \leq (\sim\frown y) \circ (\sim\frown x)$, and,
3. $x \circ y \leq z \iff x \leq \frown(y \circ \frown z)$.

Lemma

Let **A** *be a SHRA and let* $x, y, z \in A$.

1. $(x \vee y) \circ z = (x \circ z) \vee (y \circ z)$.
2. $x \circ (y \vee z) = (x \circ y) \vee (x \circ z)$.
3. $0 \circ x = x \circ 0 = 0$.

Lemma

Let **A** *be a SHRA and let* $x, y, z \in A$*.*

1. $(x \vee y) \circ z = (x \circ z) \vee (y \circ z)$.
2. $x \circ (y \vee z) = (x \circ y) \vee (x \circ z)$.
3. $0 \circ x = x \circ 0 = 0$.

Corollary

Let **A** *be a SHRA and let* I *be an ideal of* **A**. *Then* $\lambda(I)$ *is compatible with* $\circ$ *if and only if*

$$x \in I \implies (1 \circ x) \vee (x \circ 1) \in I.$$

Hence, if F *is a filter of* **A** *then* $\theta(F)$ *is a congruence if and only if*

$$x \in F \implies \neg\frown x \wedge \neg(1 \circ \sim x) \wedge \neg(\sim x \circ 1) \in F.$$

Corollary

Let $\mathcal{V}$ be a variety of SHRAs and let

$$dx = \neg{\sim}x \wedge \neg{\frown}x \wedge \neg(1 \circ {\sim}x) \wedge \neg({\sim}x \circ 1).$$

The following are equivalent:

1. *$\mathcal{V}$ is a discriminator variety.*
2. *$\mathcal{V}$ is semisimple.*
3. *$\mathcal{V}$ has EDPC and $\mathcal{V} \models x \leq d{\sim}d^m\neg x$ for some $m \in \omega$.*
4. *$\mathcal{V} \models d^{n+1}x = d^n x$ and $\mathcal{V} \models x \leq d{\sim}d^n\neg x$ for some $n \in \omega$.*