Splittings in the variety of dually pseudocomplemented Heyting algebras

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H^+ algebras

A *dually pseudocomplemented Heyting algebra* is an algebra $\mathbf{A} = \langle A; \vee, \wedge, \to, \sim, 0, 1\rangle$ such that:

- $\langle A; \vee, \wedge, \to, 0, 1\rangle$ is a Heyting algebra, and,
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Theorem

Congruences on a H^+ algebra are determined by the term $dx := \neg{\sim}x$.

What the phrase "are determined by" means is a topic for another time.

The splitting lemma

Let $\mathcal{H}^+$ denote the variety of H^+ algebras. Recall that the *diagram* of a finite H^+ algebra **A** is the term

$$\Delta_{\mathbf{A}} = \bigwedge\{[x_{a\wedge b} \leftrightarrow (x_a \wedge x_b)] \wedge [x_{a\vee b} \leftrightarrow (x_a \vee x_b)] \wedge [x_{a\to b} \leftrightarrow (x_a \to x_b)] \wedge [x_{\sim a} \leftrightarrow {\sim}x_a] \wedge [x_0 \leftrightarrow 0] \wedge [x_1 \leftrightarrow 1] \mid a, b \in A\}$$

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Lemma

Let $\mathcal{V}$ be a subvariety of $\mathcal{H}^+$ and let $\mathbf{A} \in \mathcal{V}_{fin}$ be subdirectly irreducible. The following are equivalent:

1. $\mathbf{A}$ *is not a splitting algebra in* $\mathcal{V}$*,*
2. $(\forall i \in \omega)(\exists \mathbf{B} \in \mathcal{V})\ \mathbf{A} \notin \mathcal{V}(\mathbf{B})$ *and* $\mathbf{B} \not\models d^i \Delta_{\mathbf{A}} = 0$*.*

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Lemma

Let **A**, **B** *be simple* H^+ *algebras. Then* $\mathbf{A} \in \mathcal{V}(\mathbf{B})$ *if and only if* $\mathbf{A} \not\leq \mathbf{B}$.

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Hence, condition (2) of the splitting lemma is implied by:

$$(\forall i \in \omega)(\exists \mathbf{B} \in \mathcal{V}_{\text{fsi}})\ \mathbf{A} \not\leq \mathbf{B} \text{ and } \mathbf{B} \not\models d^i \Delta_{\mathbf{A}} = 0.$$

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An ordered set X produces the algebra $\langle \mathcal{U}(X), \cup, \cap, \to, \sim, \varnothing, X\rangle$, where:

- $U \to V = X \backslash \downarrow(U \backslash V)$,
- $\sim U = \uparrow(X \backslash U)$, and,
- $d^n U = X \backslash (\downarrow\uparrow)^n (X \backslash U)$.

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So now, with $X = \mathcal{J}(\mathbf{A})^\partial$, we seek another connected ordered set Y for which there is no surjective morphism $\varphi\colon Y \to X$.

Properties of morphisms

Lemma

Let X and Y be ordered sets and let $\varphi\colon X \to Y$ be a morphism.

1. *For all $x \in X$, if x is minimal then $\varphi(x)$ is minimal and if x is maximal then $\varphi(x)$ is maximal.*
2. *For all $S \subseteq X$, if $(S, \leq_S)$ is connected then $(\varphi(S), \leq_{\varphi(S)})$ is connected.*

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and therefore $\varphi(x)$ is minimal. □

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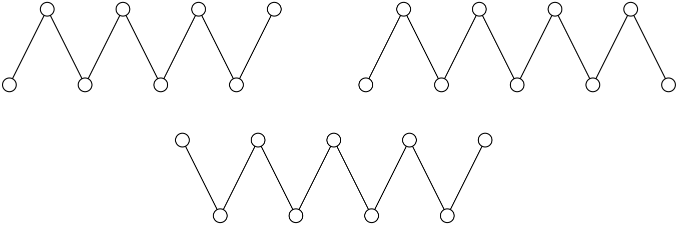
Fences



Fences

Definition

A finite ordered set X is a *fence* if there is an enumeration $x_1, \ldots, x_n$ of elements of X such that the only order relations on X are given by one of the following:

1. $x_1 < x_2 > x_3 < \cdots > x_{n-1} < x_n$,
2. $x_1 < x_2 > x_3 < \cdots < x_{n-1} > x_n$, or,
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This is not the most user-friendly definition.

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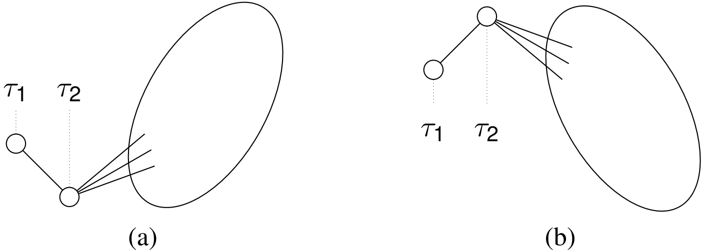

Figure: In (a), the pair (τ_1, τ_2) forms an up-tail and in (b), the pair (τ_1, τ_2) forms a down-tail.

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Let X be an ordered set and let φ be a morphism on X. If (x, y) is a down-tail in X then $(\varphi(x), \varphi(y))$ is a down-tail in $\varphi(X)$.

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Lemma

Let X be a finite connected ordered set. The following are equivalent:

1. *X is a fence.*
2. *X is has at least one tail, and $(\forall x \in X)$ $|\uparrow x| \leq 3$ and $|\downarrow x| \leq 3$.*

Extra structure

A *double-pointed ordered set* is a tuple $\mathbf{S} = \langle S; \alpha, \beta, \leq \rangle$ such that α and β are nullary operations and $\leq$ is an order on S. We will add a further assumption that α is minimal and β is maximal.

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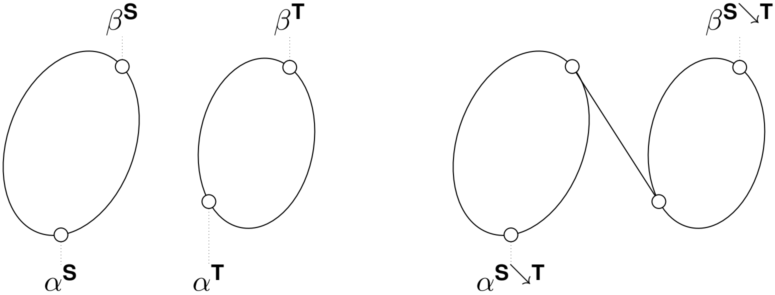
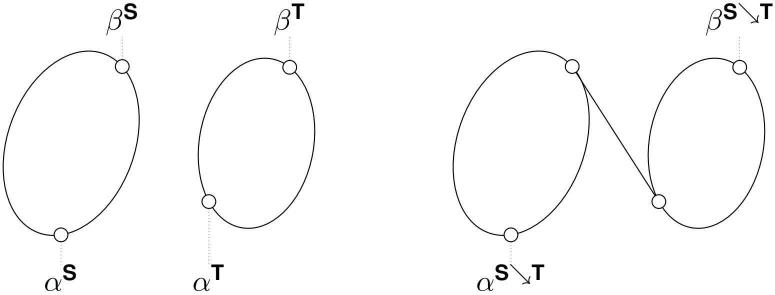
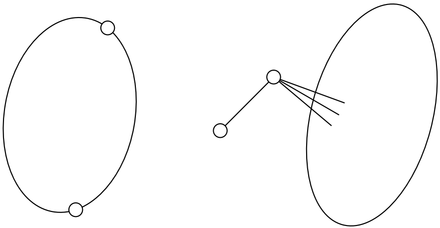
A double-pointed ordered set $\mathbf{T}$ is an *ordered set with a down-tail* if it has a down-tail (τ_1, τ_2) and $\alpha^{\mathbf{T}} = \tau_1$.

$\mathbf{S} \searrow \mathbf{T}$ illustrated



S ↘ T illustrated





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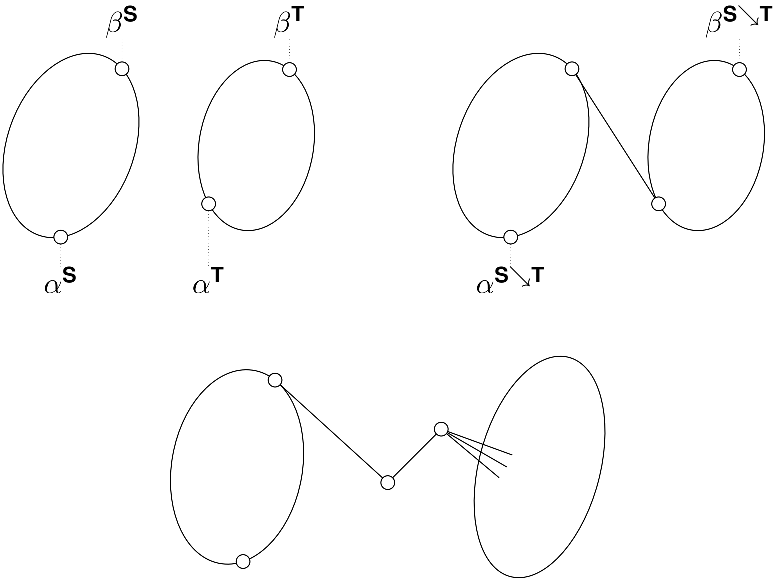

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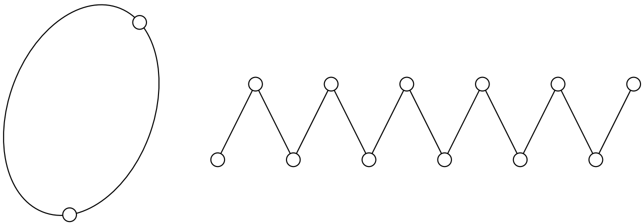

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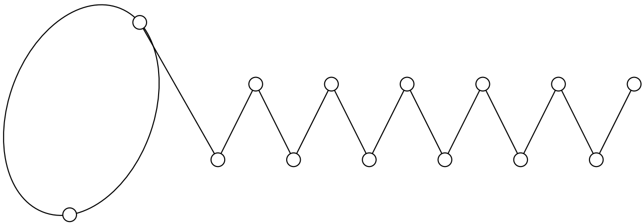

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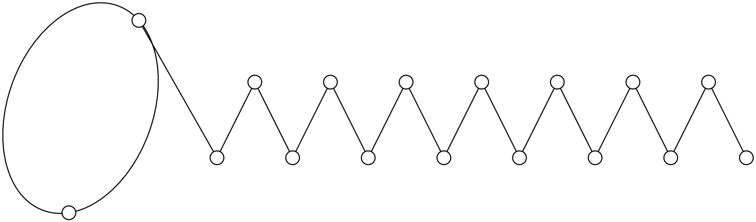

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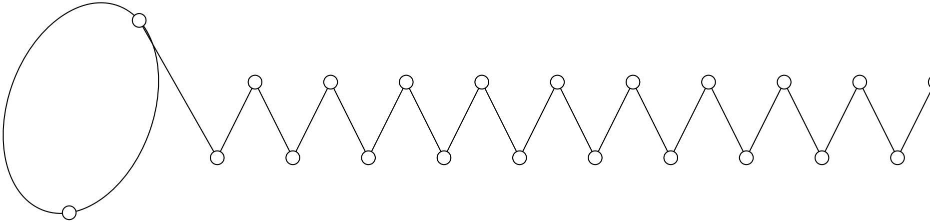

Important lemmas

Lemma 2

Let **S** *and* **T** *be double-pointed ordered sets, assume* **S** *is connected, and let* φ *be a morphism on* $\mathbf{S} \searrow \mathbf{T}$. *If* $\varphi(\beta^{\mathbf{S}}) \in \varphi(T)$, *then* $\varphi(\mathbf{S} \searrow \mathbf{T}) = \varphi(T)$.

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Let $\mathbf{S}$ *be a connected ordered set, let* $\mathbf{F}$ *be a fence with a down-tail, and let* φ *be a morphism on* $\mathbf{S} \searrow \mathbf{F}$*. If* $\varphi(\beta^{\mathbf{S}}) \notin \varphi(F)$ *then* $\varphi\restriction_F$ *is one-to-one.*

Important lemmas

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Let $\mathbf{S}$ *be a double-pointed ordered set, let* $\mathbf{F}$ *be a fence with a down-tail, and let* φ *be a morphism on* $\mathbf{S} \searrow \mathbf{F}$*. Then, for all* $x, y \in F$*:*

1. *If* $|F| > 2$ *and* (x, y) *is an up-tail in* $\mathbf{F}$ *then* $(\varphi(x), \varphi(y))$ *is an up-tail in* $\varphi(\mathbf{S} \searrow \mathbf{F})$*.*
2. *Both* $|\varphi(F) \cap {\downarrow}\varphi(x)| \leq 3$ *and* $|\varphi(F) \cap {\uparrow}\varphi(x)| \leq 3$*.*

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Corollary

Let $\mathbf{S}$ *be a connected ordered set, let* $\mathbf{F}$ *be a fence with a down-tail, and let* φ *be a morphism on* $\mathbf{S} \searrow \mathbf{F}$. *If* $\varphi\restriction_F$ *is not one-to-one, then* $\varphi(\mathbf{S} \searrow \mathbf{F})$ *is a fence.*

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$$\mathbf{X}^{(n)} = \mathbf{X}_n \searrow \mathbf{X}_{n-1} \searrow \ldots \searrow \mathbf{X}_2 \searrow \mathbf{X}_1.$$

Note that $\alpha^{\mathbf{X}^{(n)}} = (\alpha, n)$ and $\beta^{\mathbf{X}^{(n)}} = (\beta, 1)$.

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Corollary

Let **X** *be a finite double-pointed ordered set and let* **F** *be a fence with a down-tail. If* **X** *is not a fence, and* $|F| > |X|$*, then for all* $n \in \omega$ *and every morphism* φ *on* $\mathbf{X}^{(n)} \searrow \mathbf{F}$*, we have* $\varphi(\mathbf{X}^{(n)} \searrow \mathbf{F}) \not\cong X$*.*

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That is, $\mathcal{U}(\mathbf{X}) \not\leq \mathcal{U}(\mathbf{X}^{(n)} \searrow \mathbf{F})$, which is one part of the splitting lemma.

The other half

Recall that the *diagram* of a finite H^+ algebra **A** is the term

$$\Delta_{\mathbf{A}} = \bigwedge\{[x_{a\wedge b} \leftrightarrow (x_a \wedge x_b)] \wedge [x_{a\vee b} \leftrightarrow (x_a \vee x_b)] \wedge [x_{a\to b} \leftrightarrow (x_a \to x_b)] \wedge [x_{\sim a} \leftrightarrow {\sim} x_a] \wedge [x_0 \leftrightarrow 0] \wedge [x_1 \leftrightarrow 1] \mid a, b \in A\}$$

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Lemma

Let $\mathbf{X}$ and $\mathbf{Y}$ be double-pointed ordered sets and let $n \in \omega$. For each $U \in \mathcal{U}(\mathbf{X})$, map the variable $x_U \mapsto \bigcup_{i\leq n+1} U \times \{i\}$. Then, in $\mathcal{U}(\mathbf{X}^{(n+1)} \searrow \mathbf{Y})$,

$$\Delta_{\mathcal{U}(\mathbf{X})}(\overline{x}) = \bigcup_{i\leq n+1} X_i.$$

Then, $d^n(\Delta_{\mathcal{U}(\mathbf{X})}(\overline{x})) \neq \varnothing$.

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Lemma

Let X be a fence and assume $|X| > 2$. Then $\mathcal{U}(X)$ is not splitting.

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Lemma

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Proof.

If X is a fence without a down-tail, then repeat the previous slides and choose F so that it has a down-tail and apply Lemma 1.

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Okay fine, but what if $|X| = 2$?

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Okay fine, but what if $|X| = 2$?

Theorem

It splits! (But it's not easy to prove)

In full generality

Theorem

Let $\mathcal{V}$ be a variety of $\mathcal{H}^+$ algebras that is closed under $\searrow$ and contains all finite fences. Then an algebra **A** *splits the lattice of subvarieties of $\mathcal{V}$ if and only if it is the 2-element boolean algebra or the 3-element chain.*

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Further applications

- All the arguments we have seen apply to double-Heyting algebras and to congruence-regular double p-algebras.
- Let $\mathcal{H}_n^+$ (resp. $\mathcal{DH}_n$) denote the class of H^+ algebras (resp. double-Heyting algebras) whose dual space has height at most n. Each of these forms a variety, and provided that $n \geq 1$, contain all finite fences, and are closed under $\searrow$.