The Banach-Tarski Theorem

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Q Society, Wed 21 June

The Banach-Tarski Theorem

Chapter 14 of *Mathematical Conversations*, written by Robert M. French, begins with

> *It is theoretically possible, believe it or not, to cut an orange into a finite number of pieces that can then be reassembled to produce two oranges, each having exactly the same size and volume as the first one. That's right: with sufficient diligence and dexterity, from any three-dimensional solid we can produce two new objects exactly the same as the first one!*
>
> *Mathematicians, upon first hearing of this result (otherwise known as the Banach-Tarski Theorem), are generally somewhat blasé; they know that funny counter-intuitive things crop up all the time whenever infinity is involved. Most mathematicians encounter the result for the first time in graduate school and file it away in their strange results category (along with space-filling curves, Cantor functions, and non-measurable sets).*

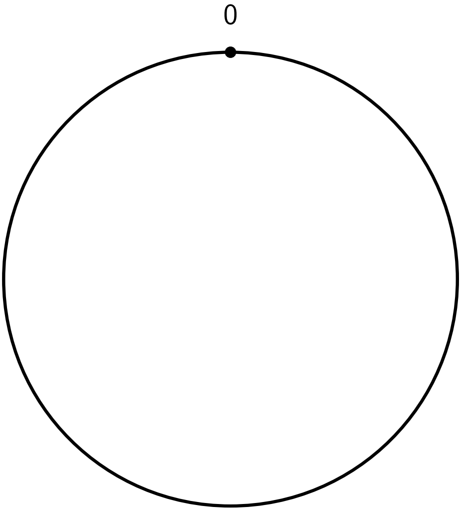
Equivalence by finite decomposition

Let X and Y be subsets of $\mathbb{R}^n$.

We say that X and Y are equivalent by finite decomposition if there exists finite partitions $\{X_1, X_2, \ldots, X_n\}$ and $\{Y_1, Y_2, \ldots, Y_n\}$ of X and Y such that X_i and Y_i are congruent for each $i \leqslant n$.

Until otherwise noted, we write $X \cong Y$ if X and Y are equivalent by finite decomposition. It is easily seen that $\cong$ is an equivalence relation.

Hilbert's merry-go-round



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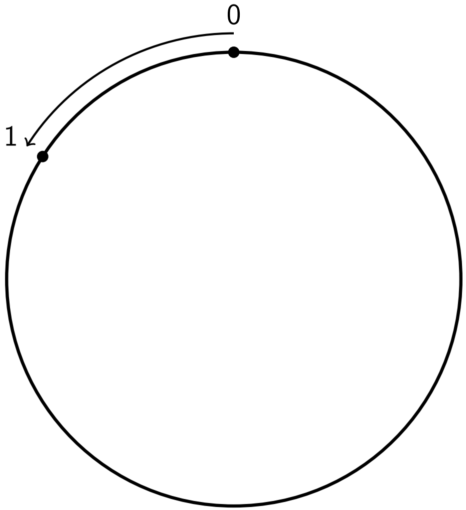

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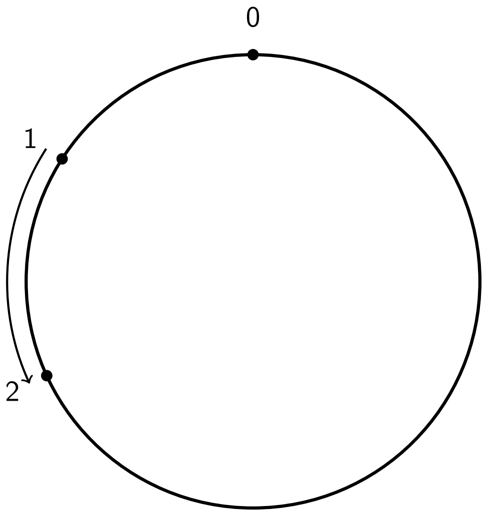

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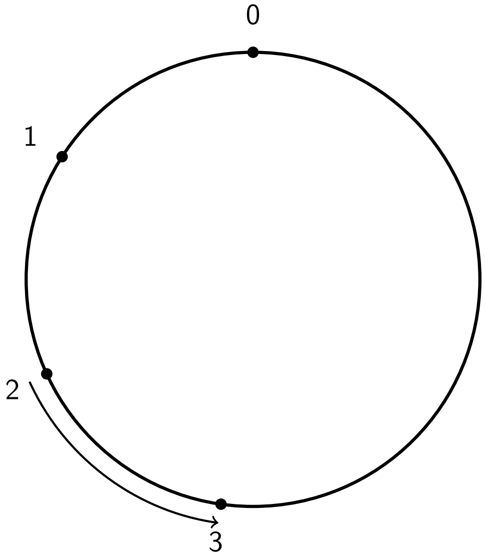

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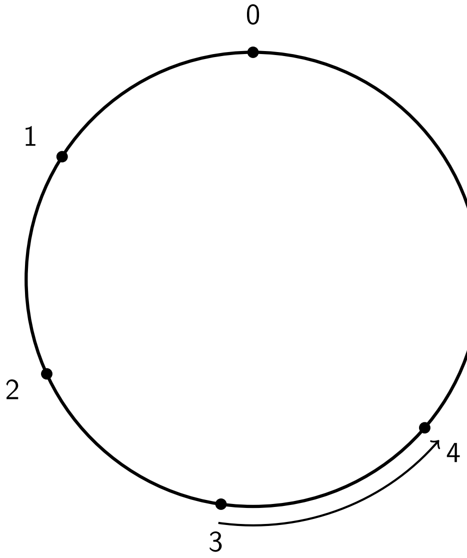

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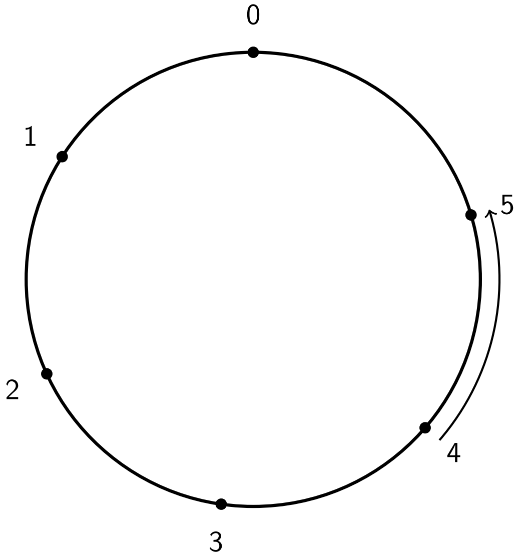

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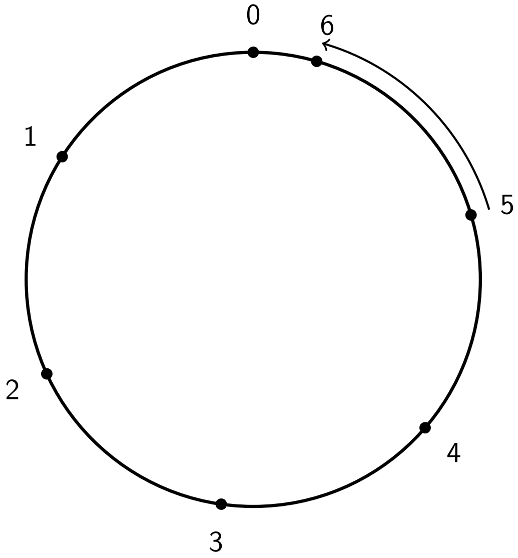

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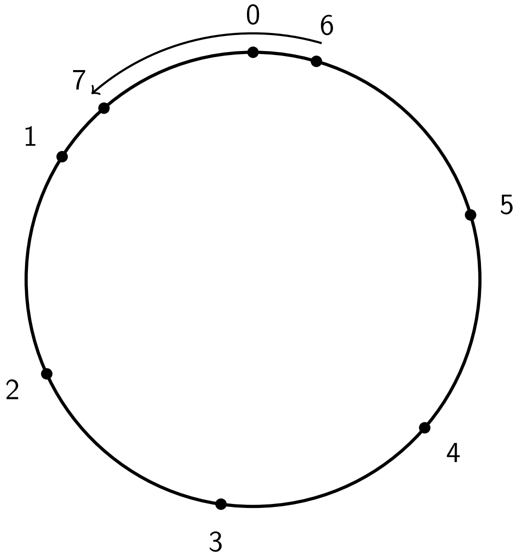

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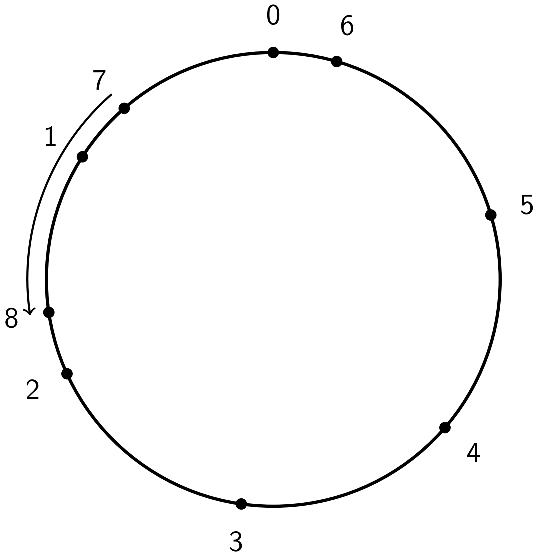

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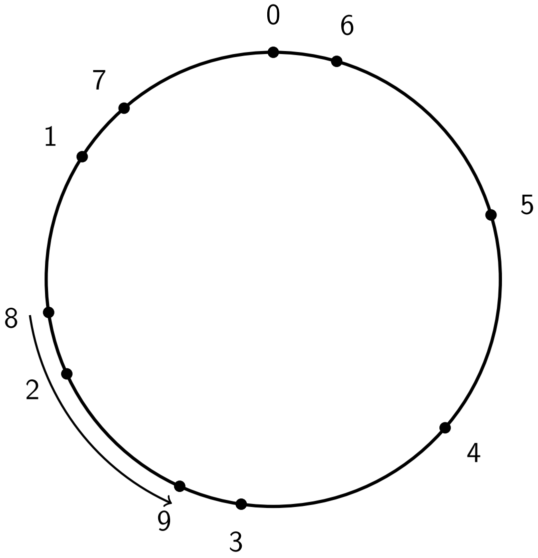

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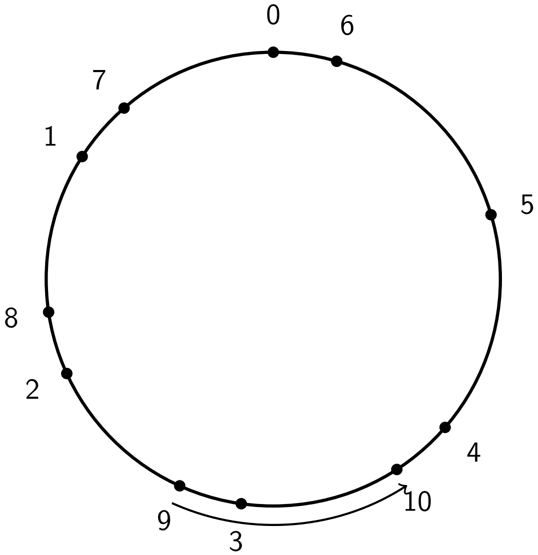

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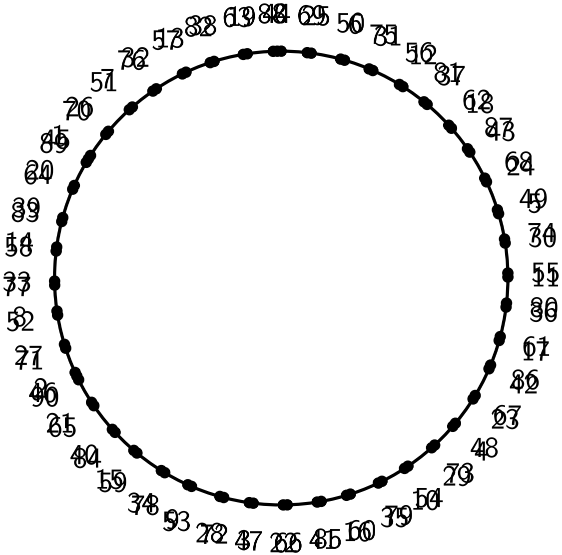

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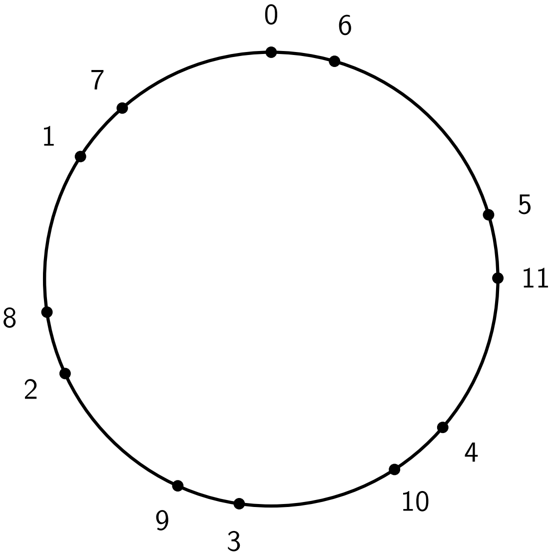

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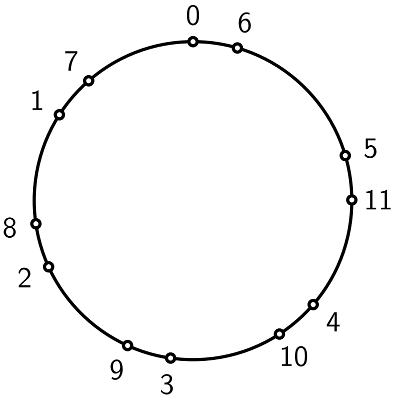

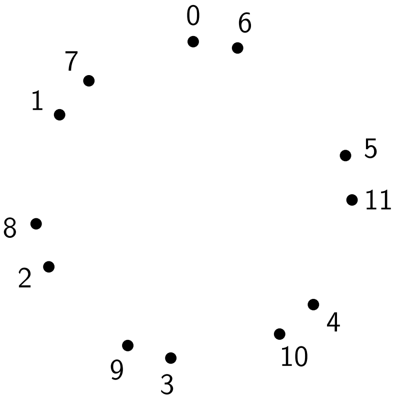

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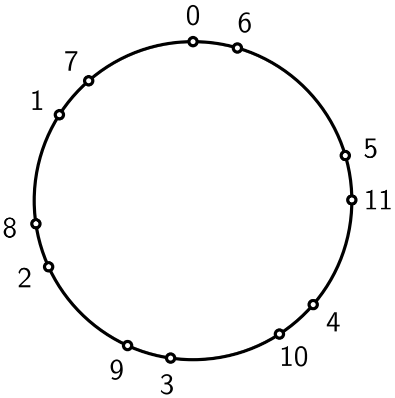

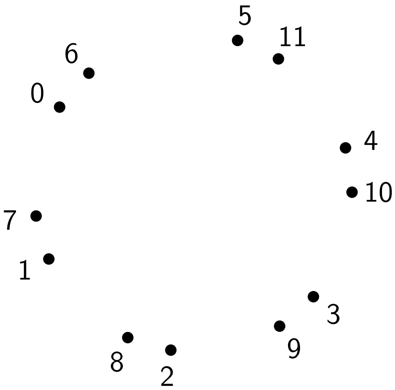

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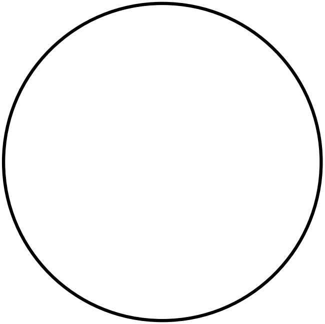

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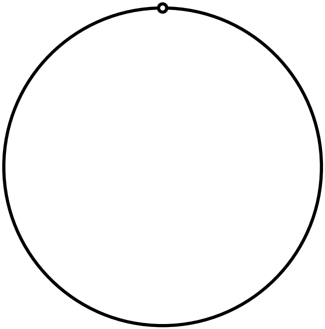
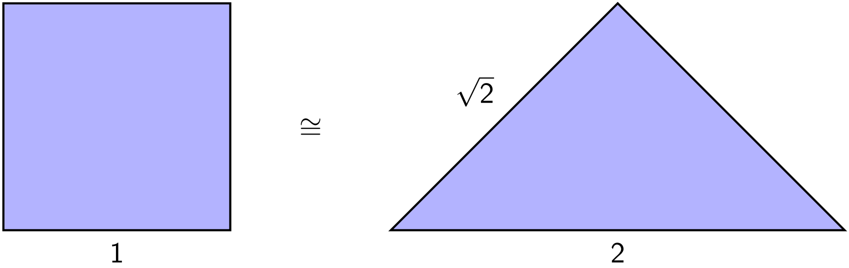


Triangulating the square

Seems obvious:



Just cut along the diagonal, right?

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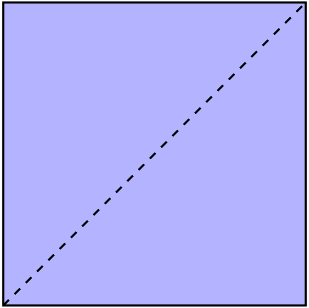

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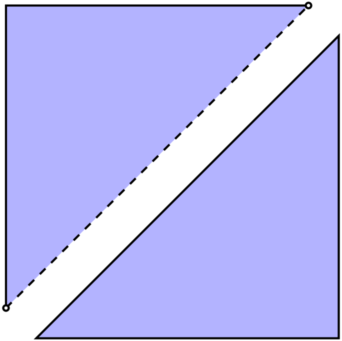

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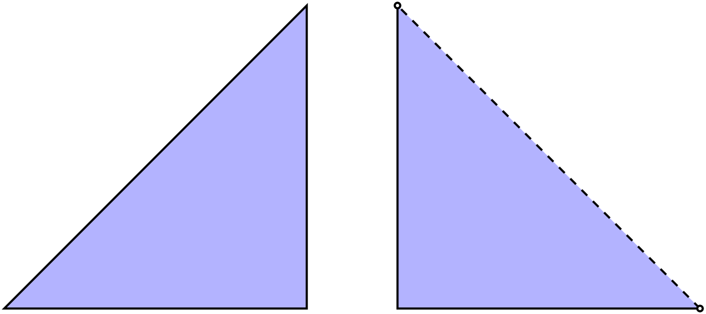

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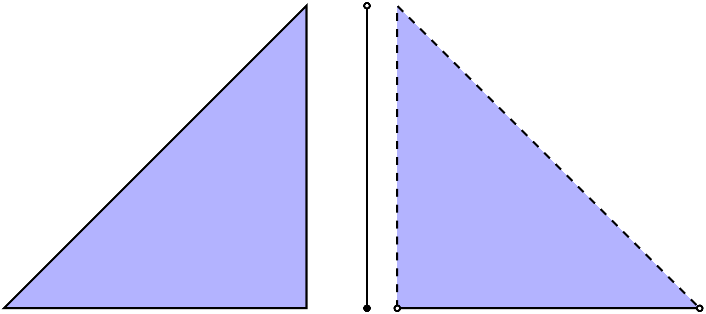

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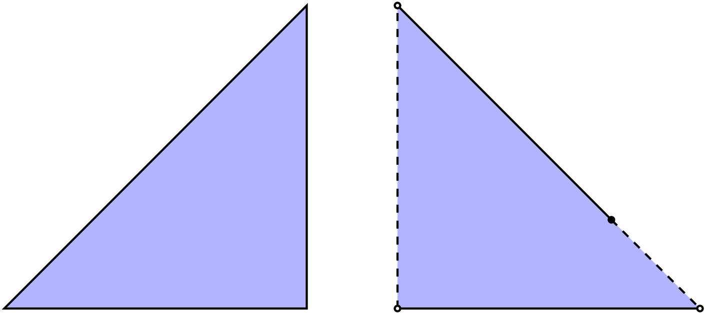

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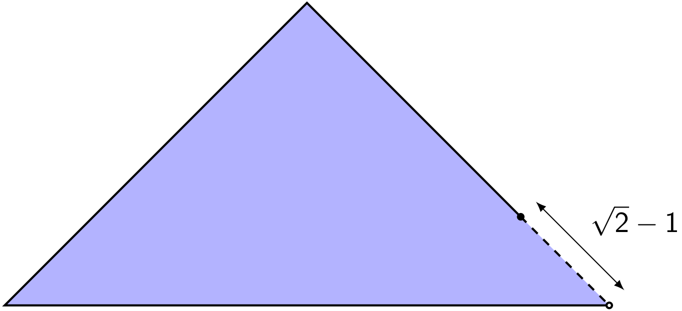

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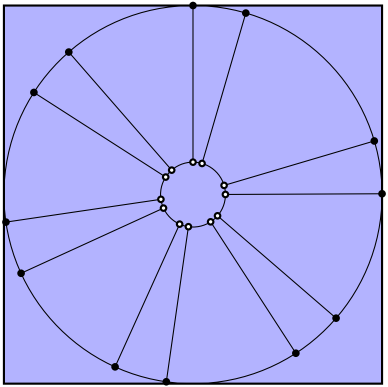

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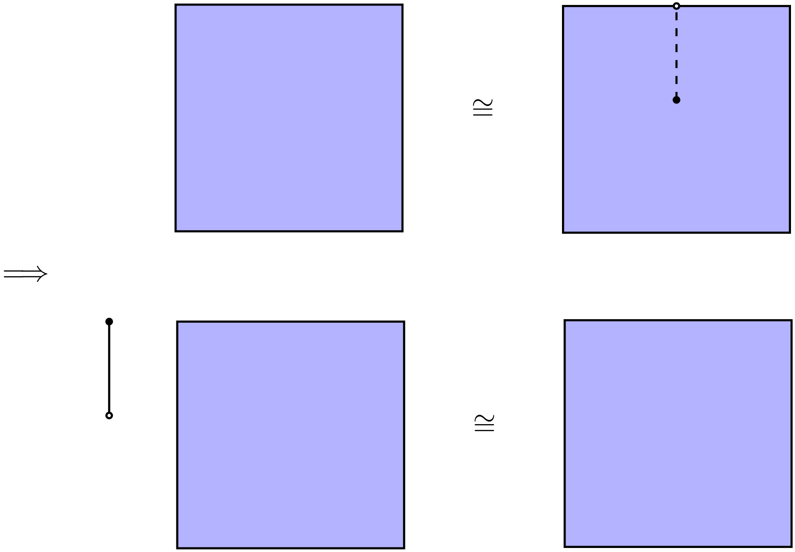

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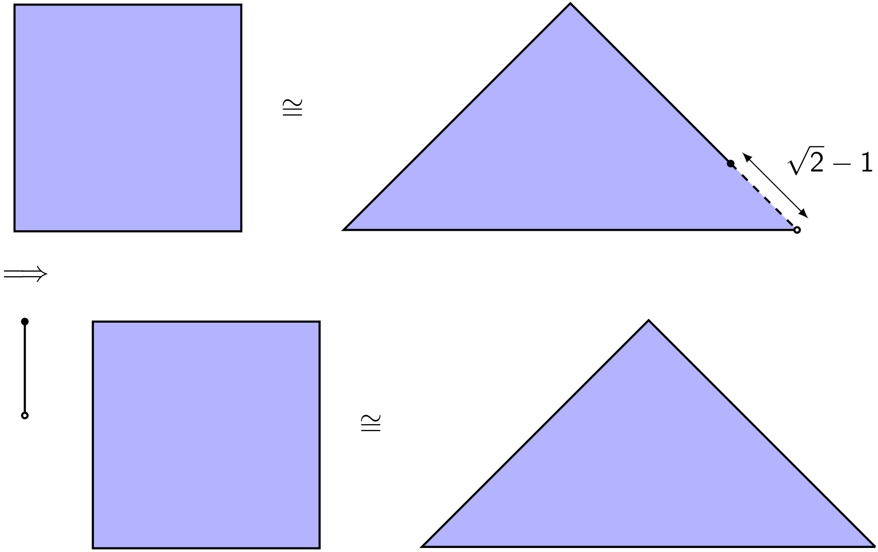
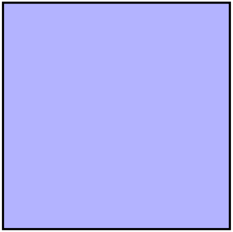


Triangulating the square

This finally gives...



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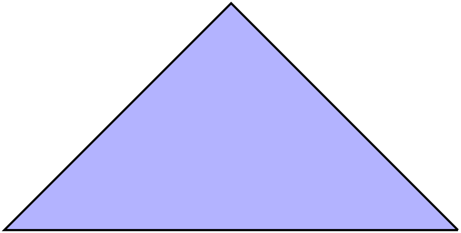

Hausdorff's paradox

At this point in the chapter, French begins a sketch of the proof of the Banach-Tarski theorem. The proof depends on *Hausdorff's paradox*.

Hausdorff's paradox
There is a countable subset D of the sphere S^2 and a partition $\{A, B\}$ of $S^2 \setminus D$ such that $S^2 \setminus D \cong A \cong B$.

Applications

In the *Applications* section, French writes

> *So we have now shown that one basketball, if it is cut up carefully enough, can spawn two. So much the better for the sports world, but what about the banking community? Can a bank note, even of the smallest denomination, produce two of its kind? Unfortunately not. The mathematician A. Lindenbaum proved that no bounded set in the plane can have a paradoxical decomposition, and a bank note, sad to say, is a bounded set in the plane.*

Generalising

Let G be a group acting on a space X and let $A, B \subseteq X$.

- We will say that A and B are congruent $(\text{mod } G)$ if there exists $g \in G$ such that $A = gB$. In that case, we write $A \equiv B \mod G$.

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- We say that A and B are (finitely) G-equidecomposable if there exist partitions $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$ of A and B, respectively, such that $A_i \equiv B_i \mod G$, for all i. In that case, we write $A \cong B \mod G$.

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- If A is non-empty, we say that A is (finitely) G-paradoxical if there are disjoint subsets $A_1, A_2 \subseteq A$ such that $A \cong A_i \mod G$, for $i \in \{1, 2\}$.

Rotations in the plane

Let C be the unit circle and let D be a countable subset of C.

We claim that

$$C \cong C \backslash D \quad \text{mod } SO(2),$$

where $SO(2)$ is the group of rotations in $\mathbb{R}^2$.

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More generally:

Lemma

Let G be a group acting on a space X, and $D \subseteq A \subseteq X$. If

- *D is countable,*
- *A is uncountable,*
- *there is a subgroup $H \leqslant G$ that acts freely on A,*

then $A \cong A \backslash D \mod G$.

Acts freely on X means $(\forall x \in X)(\forall g \in G)\ gx = x \implies g = e$.

Actions on the real line

Abuse of notation: we let $\mathbb{R}$ denote both the group $(\mathbb{R}, +)$ and the set $\mathbb{R}$.

Then $\mathbb{R}$ acts on itself.

Is there an $\mathbb{R}$-paradoxical subset of $\mathbb{R}$?

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- $SO(2)$ and $(\mathbb{R}^2, +)$ are abelian.
- $SO(2) \ltimes \mathbb{R}^2$ (formed by composing rotations and translations) is non-abelian.

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If G contains a free semigroup on 2 generators, then there exists $S \subseteq G$ such that S is G-paradoxical.

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Let f, g be the generators of the free semigroup and let S be the semigroup generated by them. That is, S is the set of words in the alphabet $\{f, g\}$. Then fS and gS are disjoint subsets of S with $S \equiv fS \equiv gS \mod G$.

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Theorem (Sierpinski–Mazurkiewicz Paradox)

There is a $SO(2) \ltimes \mathbb{R}^2$-paradoxical subset of $\mathbb{R}^2$.

- $SO(2) \ltimes \mathbb{R}^2$ contains a free semigroup S on 2 generators.
- The orbit of a point x under S is then paradoxical.

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Since $2 \times 0 = 0$, this has no conflict with doubling area.

Sherman's proof

In general, a measure satisfies *countable additivity*: for all countable collections of pairwise disjoint measurable sets $X_1, X_2, \ldots$,

$$m\left(\bigcup_{i=1}^{\infty} X_i\right) = \sum_{i=1}^{\infty} m(X_i)$$

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Solvability of $SO(2) \ltimes \mathbb{R}^2$ (and the Axiom of Choice) is sufficient.

Free groups

Proposition

The free group on two generators F_2 is finitely F_2-paradoxical.

Let a, b generate F_2. Recall that F_2 is the set of reduced words in the alphabet $\{a, a^{-1}, b, b^{-1}\}$.

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Let W_c denote the set of reduced words beginning with c. Then,

$$\{\{e\}, W_a, W_b, W_{a^{-1}}, W_{b^{-1}}\} \text{ partitions } F_2.$$

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But so do both

$$\{aW_{a^{-1}}, W_a\} \text{ and } \{bW_{b^{-1}}, W_b\}$$

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Since F_2 acts freely on X, the decomposition of F_2 can be transferred to a decomposition of X.

Hausdorff's paradox

Proposition

F_2 embeds into $SO(3)$.

Hausdorff first proved this in 1914. Hausdorff's approach is to take two rotations φ and ψ as follows:

- φ is a 180° rotation about an axis through the origin,
- ψ is a 120° rotation about an axis through the origin,
- $\cos(2\theta)$ is transcendental, where θ is the angle between the two axes.

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Osofsky's version is employed by French in Chapter 14.

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T. Tao gives an explicit generating pair in a blog post:

$$a = \frac{1}{5}\begin{pmatrix} 3 & 4 & 0 \\ -4 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad b = \frac{1}{5}\begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{pmatrix}$$

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K. Satô (1995) gives the following pair of generators:

$$a = \frac{1}{7}\begin{pmatrix} 6 & 2 & 3 \\ 2 & 3 & -6 \\ -3 & 6 & 2 \end{pmatrix}, \quad b = \frac{1}{7}\begin{pmatrix} 2 & -6 & 3 \\ 6 & 3 & 2 \\ -3 & 2 & 6 \end{pmatrix}$$

which also generate a subgroup that has no fixed points on $\mathbb{Q}^3 \cap S^2$.

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Thus we obtain Hausdorff's paradox:

Hausdorff's paradox. There is a countable subset D of the sphere S^2 such that $S^2 \backslash D$ is $SO(3)$-paradoxical.

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Then, patching up the hole in the centre can be done by translating a single point from the surface of the ball, then once again using the fact that countably many points can be ignored.

The Banach-Tarski Theorem. The closed unit ball is $SO(3) \ltimes \mathbb{R}^3$-paradoxical.

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