Double Heyting algebras vs. Boolean algebras with operators

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General Algebra Seminar
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EXPLOSIVE PALACE TELL-ALL
New Idea
WHAT MEGHAN DID...
ROYAL NANNY STORMS OUT
AFTER JUST TWO WEEKS!
CHARLES BLINDSIDED
ANNE'S FIGHT FOR THE THRONE!
MARGOT & BRAD
AMAZING PICS!
CAUGHT OUT AT CANNES
A BABY SISTER FOR LOUIS!
CUTE FAMILY ALBUM

Boolean algebras with operators

A (finite signature) Boolean algebra with operators (BAO) is an algebra

$$\mathbf{A} = \langle A; \vee, \wedge, \neg, 0, 1, f_1, f_2, \dots f_n \rangle$$

such that

- $\langle A; \vee, \wedge, \neg, 0, 1 \rangle$ is a Boolean algebra, and
- each f_i is a finitary *normal* meet-preserving operation.

Normal means that $f(\dots, 1, \dots) = 1$, and meet-preserving here is interpreted coordinatewise:

$$f(\dots, x \wedge y, \dots) = f(\dots, x, \dots) \wedge f(\dots, y, \dots).$$

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It must be noted that the definition of a BAO we are using is dual to the traditional sense. But, courtesy of the symmetry of Boolean algebras, that doesn't matter.

Relation algebras

The prototypical BAO is a *relation algebra*, first laid out by Tarski in 1948.

Definition

An algebra $\mathbf{A} = \langle A; \vee, \wedge, \circ, \smile, \neg, 0, 1, \mathrm{id} \rangle$ is a relation algebra if

- $\langle A; \vee, \wedge, \neg, 0, 1 \rangle$ is a Boolean algebra,
- $\langle A; \circ, \mathrm{id} \rangle$ is a monoid,
- $(\forall x, y, z, \in A)\ x \circ y \leqslant \neg z \Leftrightarrow \smile x \circ z \leqslant \neg y \Leftrightarrow z \circ \smile y \leqslant \neg x$.

The operation $\circ$ is called composition and $\smile$ converse.

Both $\circ$ and $\smile$ are join-preserving and 0-absorbing.

Filters

Given a Boolean algebra $\mathbf{B}$, its congruences are in one-to-one correspondence with filters of the underlying lattice; moreover, the lattice of one is isomorphic to the other:

$$\mathsf{Con}(\mathbf{B}) \cong \mathsf{Fil}(\mathbf{B})$$

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Let $x \to y = \neg x \vee y$; then let $x \leftrightarrow y = (x \to y) \wedge (y \to x)$.

It can be shown that

$$\theta(F) = \{(x, y) \in B^2 \mid x \leftrightarrow y \in F\}.$$

Congruences

Let **B** be a BAO with unary operators $f_1, \ldots, f_n$.
Given a congruence on the Boolean algebra reduct,

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what would make it a congruence on **B**?

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what would make it a congruence on $\mathbf{B}$?

Simply demand that $\theta(F)$ is compatible with each f_i:

$$x \leftrightarrow y \in F \implies f_i x \leftrightarrow f_i y \in F$$

It can be shown that this is equivalent to the demand,

$$x \in F \implies f_i x \in F.$$

Definition

Let $\mathbf{B}$ be a unary BAO with operators $f_1, \ldots, f_n$. A filter F is called a *congruence-filter* if, for all $i \leqslant n$,

$$x \in F \implies f_i x \in F.$$

Congruences

The situation for non-unary BAOs can be reduced to the unary case. For a normal operator f of arity n, define the unary operation

$$f^{(k)}(x) = f(0, \dots, 0, x, 0, \dots, 0),$$

where the x is in the k-th position.

For example, if f is 3-ary, then

$$f^{(1)}(x) = f(x, 0, 0), \quad f^{(2)}(x) = f(0, x, 0), \quad f^{(3)}(x) = f(0, 0, x)$$

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Let $\mathbf{B}$ be a BAO. Denote by $\mathbf{B}^\sharp$ the algebra obtained by replacing each operator f of $\mathbf{B}$ with the set $\{f^{(k)} \mid k \leqslant \text{arity}(f)\}$.

Proposition (Folklore)

The congruences on $\mathbf{B}$ are exactly the same as the congruences on $\mathbf{B}^\sharp$.

Congruence equivalence

Definition

Let $\mathbf{A}$ and $\mathbf{B}$ be algebras with the same underlying set but not necessarily the same signature. We will say that $\mathbf{A}$ and $\mathbf{B}$ are **congruence-equivalent** if $\mathrm{Con}(\mathbf{A}) = \mathrm{Con}(\mathbf{B})$ and write $\mathbf{A} \rightleftharpoons \mathbf{B}$.

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Thus $\mathbf{B} \rightleftharpoons \mathbf{B}^\sharp$ (where $\mathbf{B}^\sharp$ is the unary reduction).

We can reduce even further. For a unary BAO with operators $f_1, \ldots, f_n$, define the operation

$$dx := x \wedge f_1 x \wedge f_2 x \wedge \ldots f_n x,$$

and then let $\mathbf{B}^\flat = \langle B; \vee, \wedge, \neg, 0, 1, d \rangle$.
(Aside: prepending d with $x \wedge$ is a convenience that ensures $dx \leqslant x$).

Proposition (Folklore)

If $\mathbf{B}$ is a BAO, then $\mathbf{B} \rightleftharpoons \mathbf{B}^\flat$

Heyting algebras

A Heyting algebra is an algebra $\mathbf{A} = \langle A; \vee, \wedge, \rightarrow, 0, 1\rangle$ such that

- $\langle A; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice, and
- $\rightarrow$ is a binary operation satisfying

$$x \wedge y \leqslant z \iff y \leqslant x \rightarrow z.$$

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The class of Heyting algebras is an equational class defined by

1. a set of identities defining bounded distributive lattices,
2. $x \wedge (x \rightarrow y) = x \wedge y$,
3. $x \wedge (y \rightarrow z) = x \wedge [(x \wedge y) \rightarrow (x \wedge z)]$,
4. $x \wedge [(y \wedge z) \rightarrow y] = x$.

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4. $x \wedge [(y \wedge z) \to y] = x$.

We let $\neg x = x \to 0$. This is called the pseudocomplement.

Proposition

Let $\mathbf{A}$ *be a Heyting algebra. Then* $\mathrm{Con}(\mathbf{A}) \cong \mathrm{Fil}(\mathbf{A})$ *via the same map as for Boolean algebras.*

The dual pseudocomplement

An operation $\sim$ (on a bounded lattice) satisfying $x \vee y = 1 \iff y \geqslant {\sim}x$ is called a dual pseudocomplement operation.

An H^{+}-algebra is an algebra $\langle A; \vee, \wedge, \rightarrow, \sim, 0, 1\rangle$ such that

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- $\sim$ is a dual pseudocomplement operation.

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These, too, are equational. Just tack on some more equations:

1. $x \vee {\sim}(x \vee y) = x \vee {\sim}y$,
2. ${\sim}1 = 0$,
3. ${\sim}{\sim}1 = 1$.

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Theorem (Sankappanavar, 1985)

Let $\mathbf{A}$ *be an* H^+*-algebra and let* F *be a filter of* $\mathbf{A}$*. Then* F *is a congruence on* $\mathbf{A}$ *if and only if* F *is closed under* $\neg{\sim}$*.*

Double Heyting algebras

A double Heyting algebra is an algebra $\langle A; \vee, \wedge, \rightarrow, \dot{-}, 0, 1\rangle$ such that

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The class of double Heyting algebras is also equational.

The dual pseudocomplement is definable by $\sim x = 1 \dot{-} x$.

Theorem (Köhler, 1980)

Let $\mathbf{A}$ be a double Heyting algebra and F a filter of $\mathbf{A}$. Then $\theta(F)$ is a congruence on $\mathbf{A}$ if and only if F is closed under $\neg\sim$.

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Corollary

A double Heyting algebra is congruence-equivalent to its H^+*-algebra term-reduct.*

Expansions of Heyting algebras

We will say that an algebra $\mathbf{A} = \langle A; M, \vee, \wedge, \rightarrow, 0, 1\rangle$ is an expanded Heyting algebra if $\langle A; \vee, \wedge, \rightarrow, 0, 1\rangle$ is a Heyting algebra and M is a set of operations on A.

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- Let $f \colon A^n \to A$ be a map. We will say that a filter F is compatible with f if, for all $x_1, y_1, \ldots, x_n, y_n \in A$,

$$x_1 \leftrightarrow y_1, \ldots, x_n \leftrightarrow y_n \in F \implies f(x_1, \ldots, x_n) \leftrightarrow f(y_1, \ldots, y_n) \in F.$$

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- If F is compatible with every operation in M, then we say that F is compatible with M.

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- If F is compatible with every operation in M, then we say that F is compatible with M.

Recall that we define $\theta(F) = \{(x, y) \in A^2 \mid x \leftrightarrow y \in F\}$.

Proposition

$\theta(F)$ is a congruence on $\mathbf{A}$ if and only if F is compatible with M.

So we say that a filter compatible with M is a *congruence-filter*.

The fundamental problem

For an expanded Heyting algebra $\mathbf{A}$, find a unary term t in the language of $\mathbf{A}$ such that

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A term meeting this condition will be called a *congruence-filter term on* **A** or a *compatability term for* M.

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Proposition

If t_1 *is a compatability term for* M_1 *and* t_2 *is a compatability term for* M_2*, then the term* t *defined by* $tx = t_1x \wedge t_2x$ *is a compatability term for* $M_1 \cup M_2$*.*

Therefore, if M is finite, it would suffice to find a compatability term for each element of M.

Heyting algebras with operators

Lemma 1 (Hasimoto, 2001)

Let $\mathbf{A}$ *be a Heyting algebra, let* f *be a unary map on* A *and let* F *be a filter on* $\mathbf{A}$*. If* f *is a normal operator, then* F *is compatible with* f *if and only if* F *is closed under* f*.*

Proof.

($\Rightarrow$) As $x \leftrightarrow 1 = x$, if $x \in F$ then $fx = fx \leftrightarrow 1 = fx \leftrightarrow f1 \in F$.
($\Leftarrow$) First note that,

$$fb \geqslant fa \wedge fb = f(a \wedge b)$$

□

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Thus, by definition of $\to$, we have $f(a \to b) \leqslant fa \to fb$

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Thus, by definition of $\rightarrow$, we have $f(a \rightarrow b) \leqslant fa \rightarrow fb$, and similarly, $f(b \rightarrow a) \leqslant fb \rightarrow fa$.

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Thus, by definition of $\to$, we have $f(a \to b) \leqslant fa \to fb$, and similarly, $f(b \to a) \leqslant fb \to fa$. Now, if $a \leftrightarrow b \in F$, then $a \to b \in F$ and $b \to a \in F$.

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Thus, by definition of $\to$, we have $f(a \to b) \leqslant fa \to fb$, and similarly, $f(b \to a) \leqslant fb \to fa$. Now, if $a \leftrightarrow b \in F$, then $a \to b \in F$ and $b \to a \in F$. Because F is closed under f, we then have $f(a \to b) \in F$ and $f(b \to a) \in F$.

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Lemma 1 (Hasimoto, 2001)

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Thus, by definition of $\to$, we have $f(a \to b) \leqslant fa \to fb$, and similarly, $f(b \to a) \leqslant fb \to fa$. Now, if $a \leftrightarrow b \in F$, then $a \to b \in F$ and $b \to a \in F$. Because F is closed under f, we then have $f(a \to b) \in F$ and $f(b \to a) \in F$. It then follows that $fa \to fb \in F$ and $fb \to fa \in F$, and hence $fa \leftrightarrow fb \in F$. Hence, F is compatible with f. □

Normalising

The following construction is due to Hasimoto (2001).

Definition

Let $\mathbf{A}$ be a Heyting algebra and let $f\colon A^n \to A$ be a map. For each $a \in A$, define the set $f^{\leftrightarrow}(a)$ by

$$f^{\leftrightarrow}(a) := \{f(x_1, \dots, x_n) \leftrightarrow f(y_1, \dots, y_n) \mid (\forall i \leqslant n)\ x_i \leftrightarrow y_i \geqslant a\}.$$

For any set K of maps on A, let $[K]\colon A \to A$ be the partial operation

$$[K]a = \bigwedge \bigcup \{f^{\leftrightarrow}(a) \mid f \in K\}.$$

We say that $[K]$ *exists in* $\mathbf{A}$ if it is defined for all $a \in A$. If $K = \{f\}$ we will write $[f]$ instead.

Normalising

Lemma 2 (Hasimoto, 2001)

Let $\mathbf{A}$ *be a Heyting algebra and let* K *be a set of operations on* A*. If* $[K]$ *exists, then* $[K]$ *is a normal operator, and* $[[K]] = [K]$*.*

Definition

Let $\mathbf{A}$ be an expanded Heyting algebra and let M denote the set of operations on $\mathbf{A}$. Let $[\mathbf{A}]$ denote the algebra

$$\langle A; \vee, \wedge, \rightarrow, [M], 0, 1 \rangle.$$

Theorem 3 (Hasimoto, 2001)

Let $\mathbf{A}$ *be an expanded Heyting algebra and assume that* $[M]$ *exists in* $\mathbf{A}$*.*

1. $\mathrm{CFil}([\mathbf{A}]) \subseteq \mathrm{CFil}(\mathbf{A})$.
2. $\mathbf{A} \rightleftharpoons [\mathbf{A}]$ *if and only if* $(\forall a \in A)\ [M]a \in \mathrm{CFg}^{\mathbf{A}}(a)$.

Proof.

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For part (2), the condition $(\forall a \in A)\ [M]a \in \mathrm{CFg}^{\mathbf{A}}(a)$ is equivalent to the claim that every congruence-filter of $\mathbf{A}$ is closed under $[M]$. Part (2) then holds after a brief explanation. □

Getting a term

Lemma

Let $\mathbf{A}$ *be an expanded Heyting algebra and assume* $[M]$ *exists in* $\mathbf{A}$*. If there is a term* t *in the language of* $\mathbf{A}$ *such that* $tx = [M]x$*, then* t *is a normal filter term on* $\mathbf{A}$*.*

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If F is closed under t, then it is a congruence-filter of $[\mathbf{A}]$. Then by Theorem 3, F is a congruence-filter of $\mathbf{A}$.

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Although nice, Hasimoto's construction does not have to produce a term function on $\mathbf{A}$, even when it exists.

Some existence conditions

Definition

Let $\mathbf{A}$ be a Heyting algebra and let $f\colon A^n \to A$ be a map. For each $k \leqslant n$, let $f^k\colon A \to A$ denote the unary map given by

$$f^{(k)}x = f(0, \ldots, 0, x, 0, \ldots, 0),$$

where x is in the k-th position. We will call f an *operator* if:

1. $f(\ldots, x \wedge y, \ldots) = f(\ldots, x, \ldots) \wedge f(\ldots, y, \ldots)$, and,
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Lemma (Hasimoto, 2001)

Let $\mathbf{A}$ *be a Heyting algebra and let* $f\colon A^n \to A$ *be an operator. Then* $[f]$ *exists, and*

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Lemma

Let **A** *be a Heyting algebra and let* f *be an anti-operator. Then* $[f]$ *exists, and*

$$[f]x = \bigwedge_{k \leqslant n} \neg f^{(k)} x.$$

An example

Recall that a unary operation $\sim$ is a dual pseudocomplement operation if it satisfies the equivalence $x \vee y = 1 \iff y \geqslant {\sim}x$.

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and so ${\sim}x \vee {\sim}y \geqslant {\sim}(x \wedge y)$. □

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Corollary

Let $\mathbf{A}$ *be a Heyting algebra and let* $\sim$ *be a dual pseudocomplement operation on* $\mathbf{A}$. *Then* $[\sim] = \neg\sim$.

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*Let **A** be a Heyting algebra and let $\sim$ be a dual pseudocomplement operation on **A**. Then $[\sim] = \neg\sim$.*

Corollary (Sankappanavar, 1985)

*Let **A** be an H^{+}-algebra. Then $\neg\sim$ is a congruence-filter term on **A**.*

We also saw

Theorem (Köhler, 1980)

*Let **A** be a double Heyting algebra and F a filter of **A**. Then $\theta(F)$ is a congruence on **A** if and only if F is closed under $\neg\sim$.*

We will obtain this result as well by showing that $[\dot{-}] = \neg\sim$.

Double Heyting algebras

Proof that $[\dot{-}] = \neg\sim$.

By definition, we have $[\dot{-}]a = \bigwedge \dot{-}^{\leftrightarrow}(a)$, where

$$\dot{-}^{\leftrightarrow}(a) = \{(x_1 \dot{-} x_2) \leftrightarrow (y_1 \dot{-} y_2) \mid x_i \leftrightarrow y_i \geqslant a\}.$$

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(1) Since $1 \leftrightarrow 1 = 1 \geqslant a$ and $a \leftrightarrow 1 = a$, it follows that

$$(1 \dot{-} a) \leftrightarrow (1 \dot{-} 1) \in \dot{-}^{\leftrightarrow}(a)$$

But $1 \dot{-} 1 = 0$ and $1 \dot{-} a = \sim a$, so $\sim a \leftrightarrow 0 = \neg\sim a \in \dot{-}^{\leftrightarrow}(a)$. □

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(2) Note that $x \leftrightarrow y \geqslant z$ if and only if $x \wedge z = y \wedge z$. So we will prove that, if $x_i \leftrightarrow y_i \geqslant a$, then $(x_1 \dot{-} x_2) \wedge \neg\sim a = (y_1 \dot{-} y_2) \wedge \neg\sim a$. We have,

$$\begin{aligned} x_i \leftrightarrow y_i \geqslant a &\implies x_i \wedge a = y_i \wedge a \\ &\implies (x_i \wedge a) \vee \sim a = (y_i \wedge a) \vee \sim a \\ &\implies x_i \vee \sim a = y_i \vee \sim a. \end{aligned}$$

The equation $x \vee (y \dot{-} z) = x \vee [(y \vee x) \dot{-} (z \vee x)]$ holds in all double Heyting algebras. So,

$$\begin{aligned} \sim a \vee (x_1 \dot{-} x_2) &= \sim a \vee [(x_1 \vee \sim a) \dot{-} (x_2 \vee \sim a)] \\ &= \sim a \vee [(y_1 \vee \sim a) \dot{-} (y_2 \vee \sim a)] \\ &= \sim a \vee (y_1 \dot{-} y_2). \end{aligned}$$

Taking a meet with $\neg\sim a$ will finish the job. □