Constructible Numbers and Origami

Christopher J. Taylor

Compass and straightedge

Given a pair of existing points, the following may be constructed:

1. the line through any two different existing points,
2. the circle with center at one point and through another point,
3. the point which is the intersection of two lines
4. the points which are on the intersection of a line and circle
5. the points which are on the intersection of two circles

Item 3 may have 0 or 1 solutions; items 4 and 5 may have 0, 1 or 2.

Compass and straightedge

Definition

A real number x is *constructible* if $|x|$ is the distance between two points constructed as above. An angle θ is constructible if $\cos(\theta)$ is a constructible real number. Let $\mathcal{C}$ denote the set of constructible reals.

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In particular, a quadratic with constructible coefficients and real solutions has constructible solutions.

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Let $\theta = \arccos(a)$ be given; we aim to construct $\theta/2$; i.e., the length $\cos(\theta/2) = \cos(\arccos(a)/2)$.

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Thus, assuming constructibility of a ensures constructibility of $\cos(\arccos(a)/2)$.

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- squaring the circle,
- doubling the cube,
- trisecting the angle.

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For a constructible number x:

- Squaring the circle amounts to constructing $x \cdot \sqrt{\pi}$.
- Doubling the cube amounts to constructing $x \cdot \sqrt[3]{2}$.
- Trisecting the angle amounts to constructing $\cos(\arccos(x)/3)$.

Impossible constructions

Theorem

A real number x is constructible if and only if there is a chain of algebraic field extensions

$$\mathbb{Q} = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n,$$

where $x \in K_n$ and $[K_{j+1} : K_j] = 2$, for each $0 \leqslant j < n$.

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If $a \in \mathbb{Q}$ but $\sqrt[3]{a} \notin \mathbb{Q}$, the minimal polynomial of $\sqrt[3]{a}$ is $x^3 - a$. Thus cube roots are not constructible (unless $\sqrt[3]{a}$ is rational).

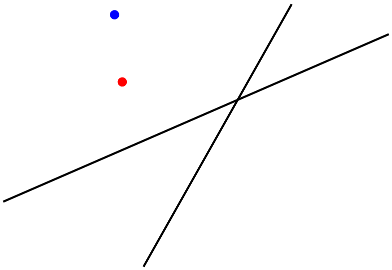
Margherita Piazzola Beloch (1879–1976, right)

Her main scientific interest were algebraic geometry, algebraic topology and photogrammetry. After her thesis she worked on classification of algebraic surfaces studying the configurations of lines that could lie on surfaces. The next step was to study rational curves lying on surfaces and in this framework Beloch obtained the following important result: "Hyperelleptic surfaces of rank 2 are characterised by having 16 rational curves." — Wikipedia entry

The Beloch Fold

In 1936, Beloch showed that origami was suitable for doubling the cube by utilising what is now called the Beloch fold.

Given two points P_1, P_2 and two lines ℓ_1, ℓ_2 suitably positioned, you can create a fold that takes P_1 onto ℓ_1 and P_2 onto ℓ_2.



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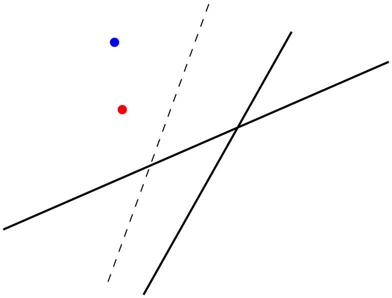

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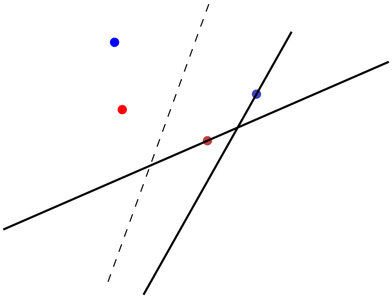
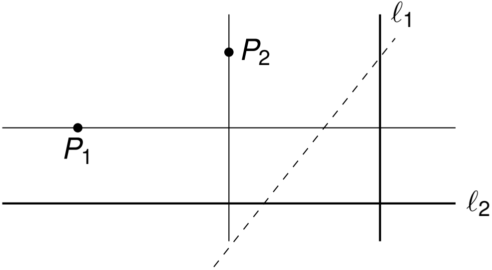


The Beloch Fold

The cube root construction assumes the existence of two points, $P_1 = (-r, 0)$ and $P_2 = (0, 1)$.

Take the line ℓ_2 given by $y = -1$ and ℓ_1 given by $x = r$. Perform the Beloch fold, folding P_2 onto ℓ_2 and P_1 onto ℓ_1, as shown.

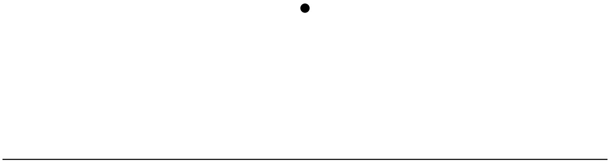


Claim: the x-intercept of the fold is $\sqrt[3]{r}$.

This can be proved geometrically, but we will take a different approach.

Folding

Consider the action of just folding a point P onto a line L.



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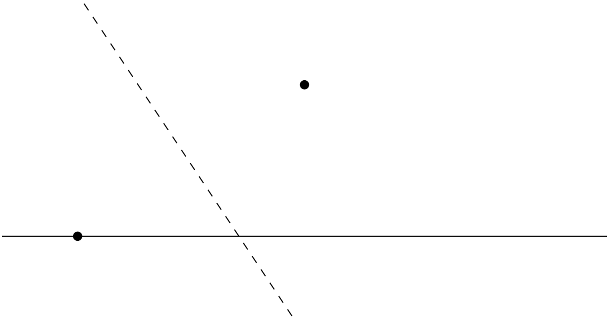

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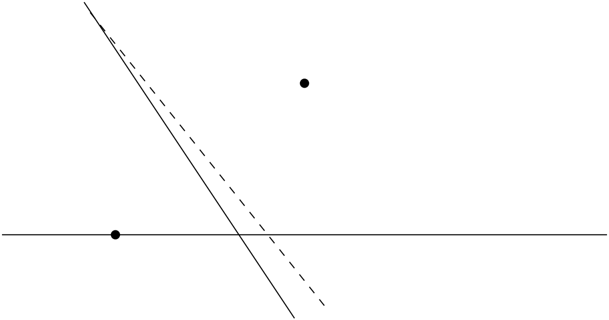

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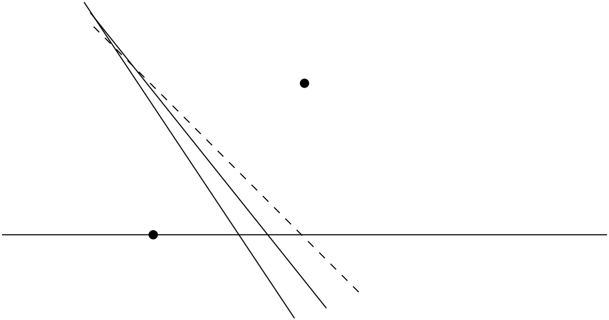

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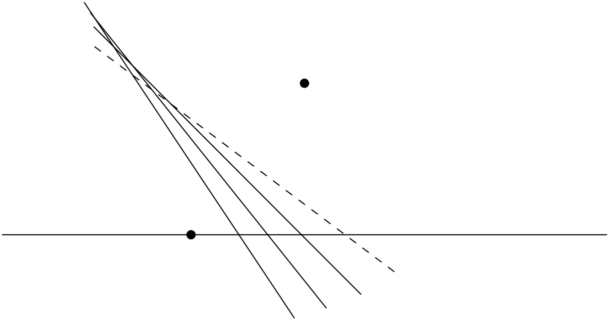

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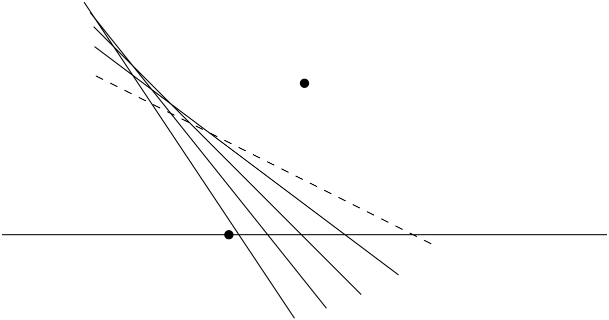

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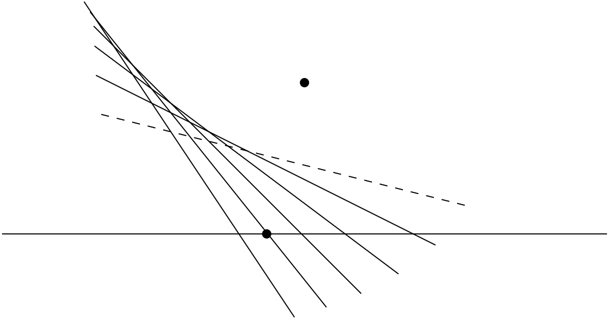

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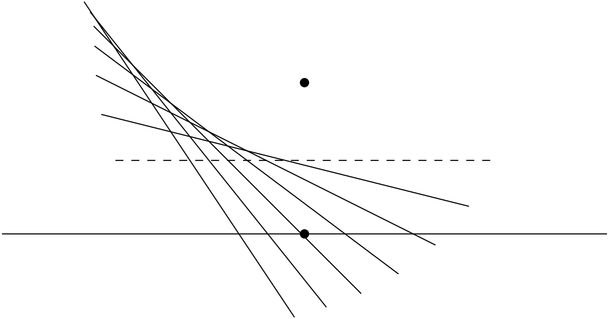

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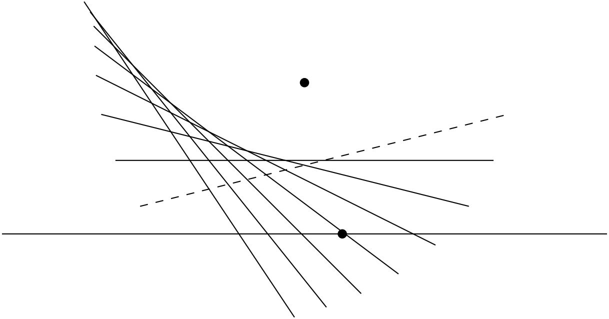

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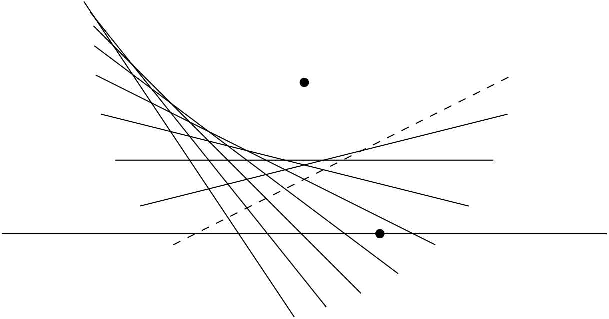

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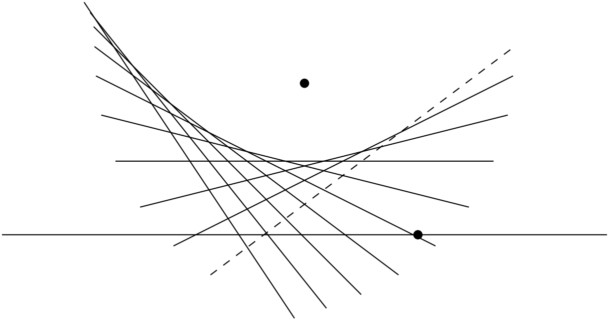

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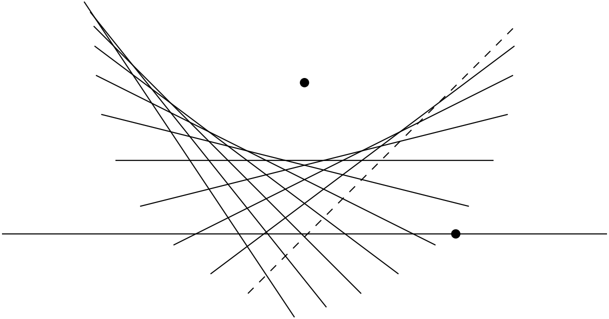

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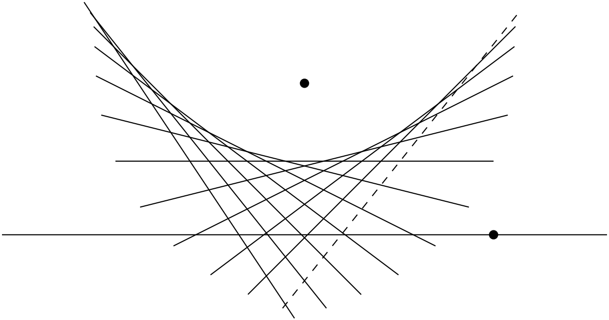

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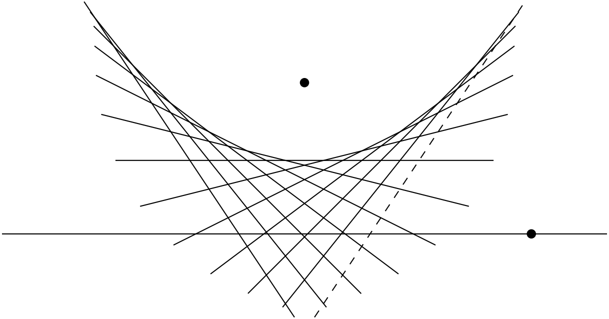

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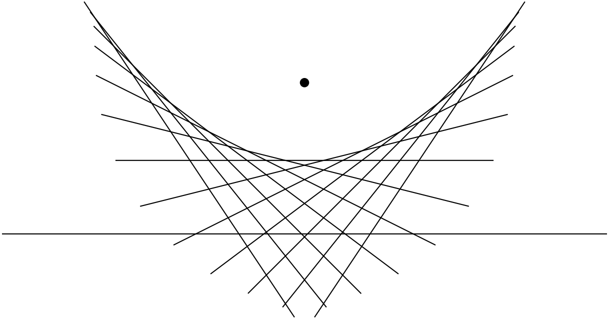

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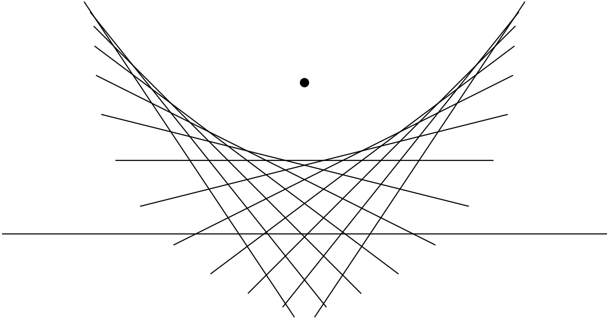

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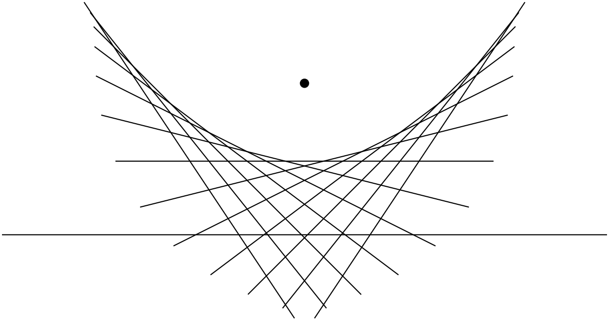

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Consequently, finding a solution for the Beloch fold amounts to finding a tangent line common to two parabolas.

Parabolas

Given focus (f_1, f_2) and directrix $ax + by + c = 0$, the equation for the associated parabola is

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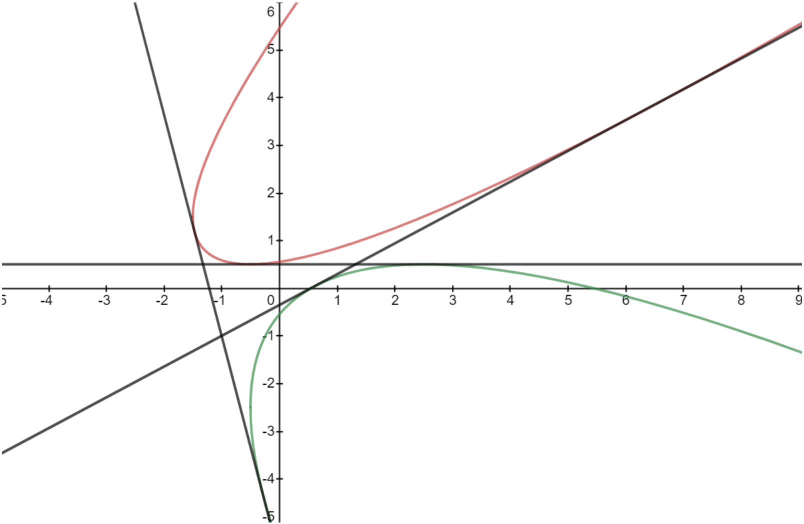
For parabolas, one of these is the tangent at infinity. Thus, in general, there are at most three solutions for the Beloch fold.

Parabolas



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We want a common tangent of the two parabolas p_1 and p_2, where the equations are, respectively,

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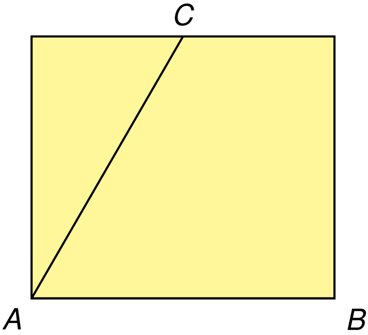
Trisecting the angle

Now, given an angle *CAB* as shown below, we can trisect it as follows:



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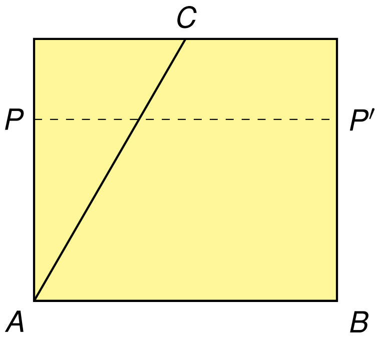

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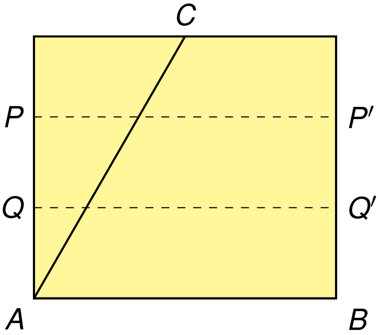

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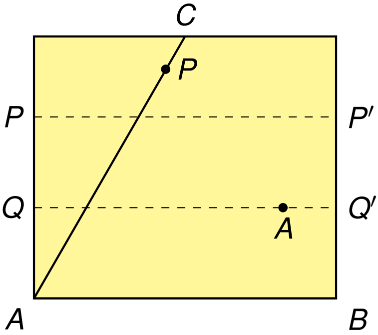

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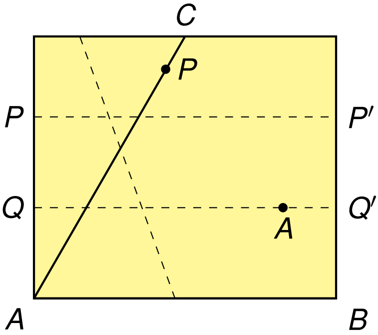

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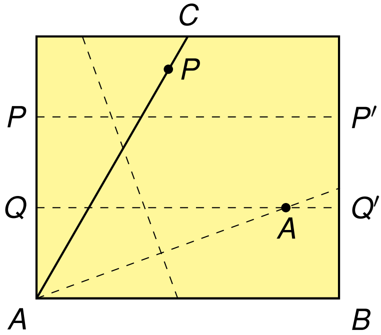
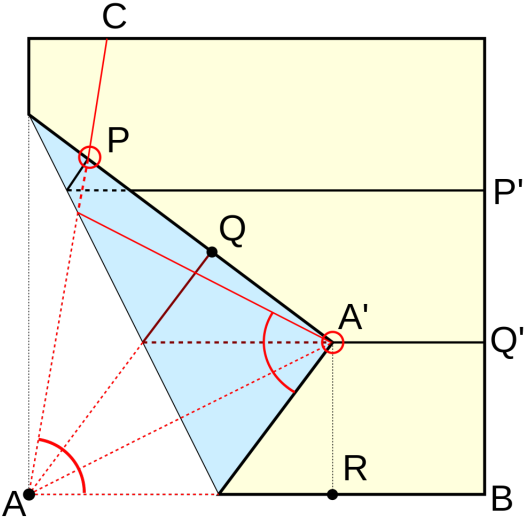


- Fold a line PP' perpendicular to AB.
- Fold AB onto PP'; call this fold QQ'.
- Fold P onto AC and A onto QQ'.
- Fold this new line onto itself such that A is a fixed point.

Trisecting the angle

Nicked from Wikipedia:



Axioms of origami

1. Given two distinct points P_1 and P_2, construct the line that passes through both of them.
2. Given two distinct points P_1 and P_2, construct the line that places P_1 onto P_2.
3. Given two lines ℓ_1 and ℓ_2, construct the line that places ℓ_1 onto ℓ_2.
4. Given a point P_1 and a line ℓ_1, construct the line perpendicular to ℓ_1 that passes through P_1.
5. Given two points P_1, P_2 and a line ℓ_1, construct a line that places P_1 onto L_1 and passes through P_2
6. (The Beloch Fold) Given two points P_1, P_2 and two lines ℓ_1, ℓ_2, construct a line that takes P_1 onto ℓ_1 and P_2 onto ℓ_2.
7. Given a point P and two lines ℓ_1 and ℓ_2, construct a line that places P onto ℓ_1 and is perpendicular to ℓ_2.

These axioms are known as the Huzita-Hatori axioms.

Constructibility

A point in origami geometry is the intersection of two lines.

Definition

A real number x is origami-constructible if $|x|$ is the distance between two points constructed using the axioms on the previous slide.

Denote by $\mathcal{O}$ the set of origami-constructible numbers.

Theorem

Axioms 1–5 are essentially the same as the compass and straightedge axioms.

Corollary

$\mathcal{O}$ is a field; moreover, $\mathcal{C}$ is a subfield of $\mathcal{O}$.

Constructibility

Theorem

A real number x is origami-constructible if and only if there is a chain of algebraic field extensions

$$\mathbb{Q} = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n,$$

where $x \in K_n$ and $[K_{j+1} : K_j] \in \{2, 3\}$, for each $0 \leqslant j < n$.

The bulk of the proof amounts to showing that the Beloch fold is equivalent to solving a cubic equation with origami-constructible coefficients.