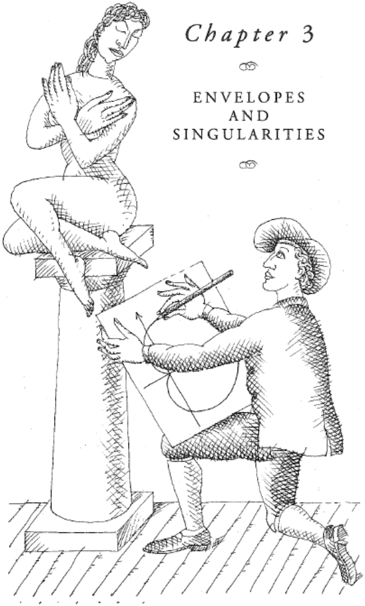
Lecture 9: Cusps

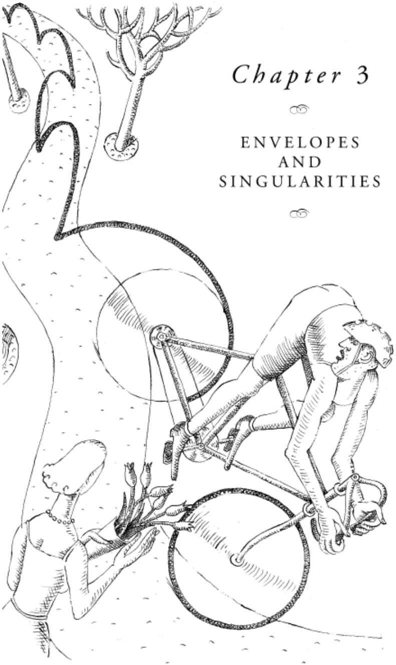
Christopher J. Taylor

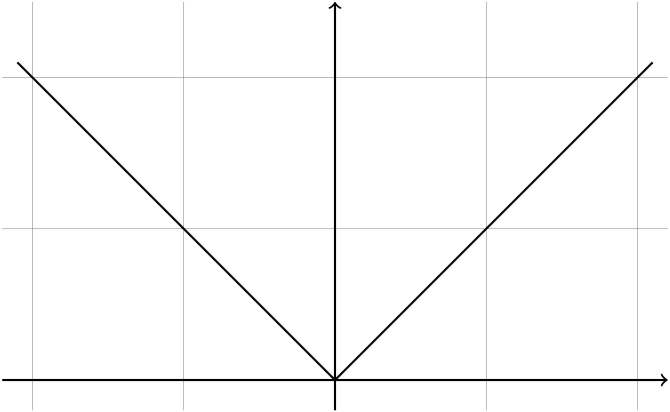
La Trobe University

Q-Society
May 2, 2018

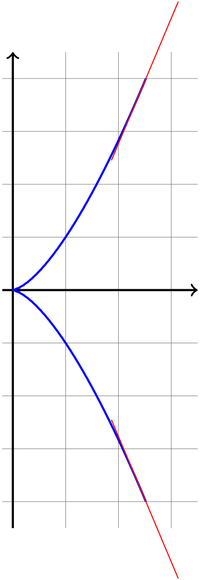

Chapter 3

ENVELOPES
AND
SINGULARITIES

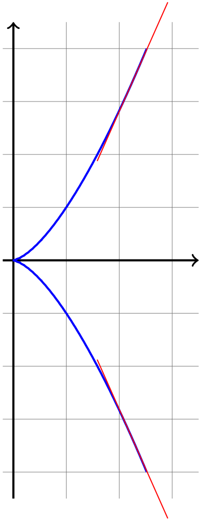

Chapter 3
ENVELOPES
AND
SINGULARITIES



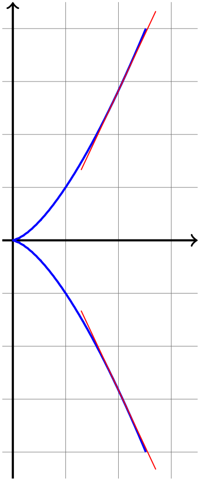
$$y = |x|$$



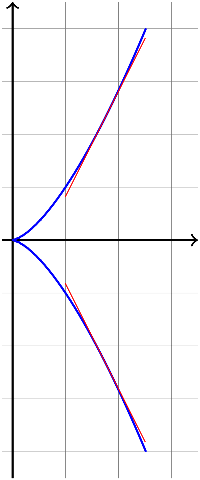
$$x^2 = y^3$$



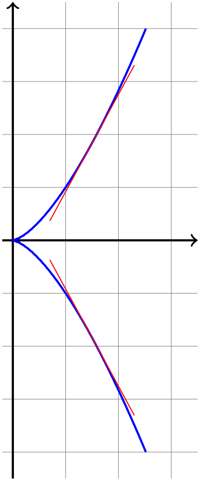
$$x^2 = y^3$$



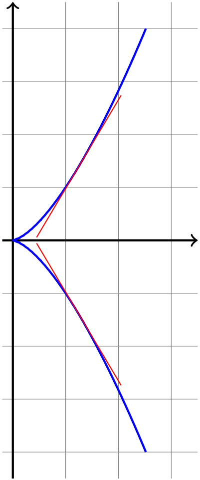
$$x^2 = y^3$$



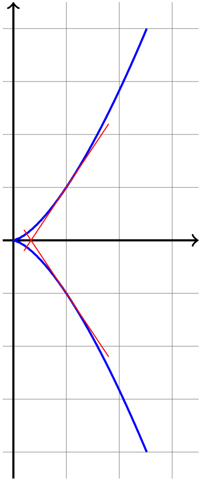
$$x^2 = y^3$$



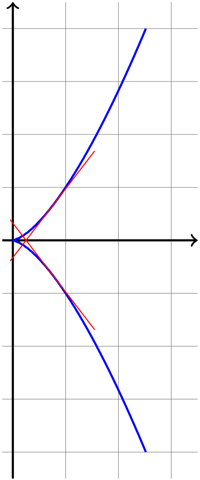
$$x^2 = y^3$$



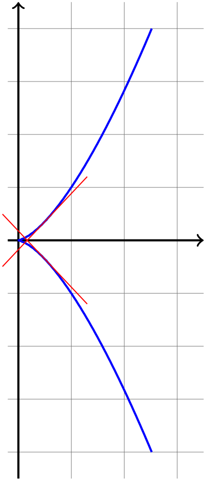
$$x^2 = y^3$$



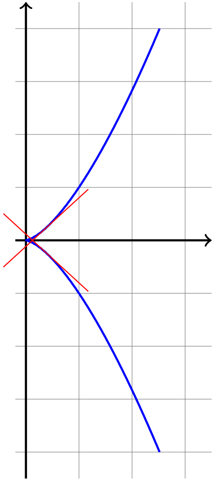
$$x^2 = y^3$$



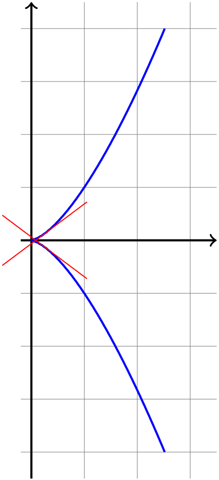
$$x^2 = y^3$$



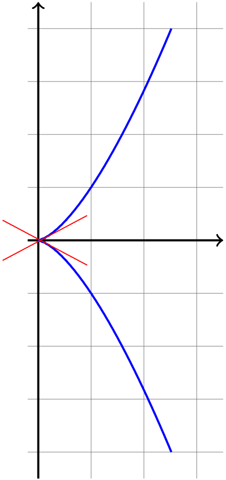
$$x^2 = y^3$$



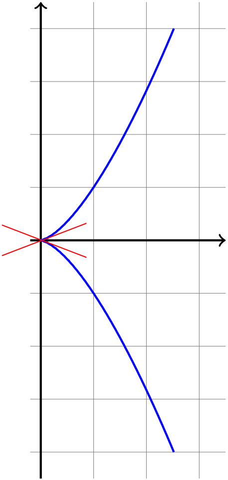
$$x^2 = y^3$$



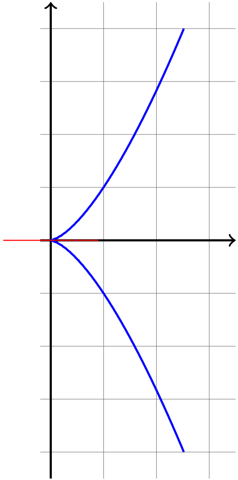
$$x^2 = y^3$$



$$x^2 = y^3$$

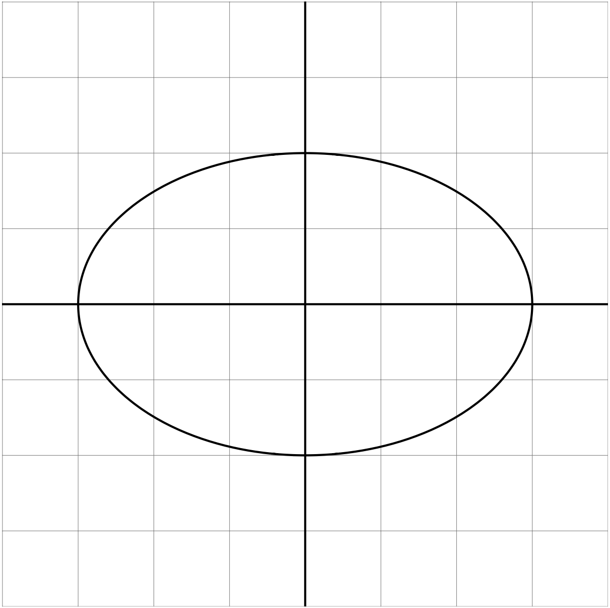


$$x^2 = y^3$$

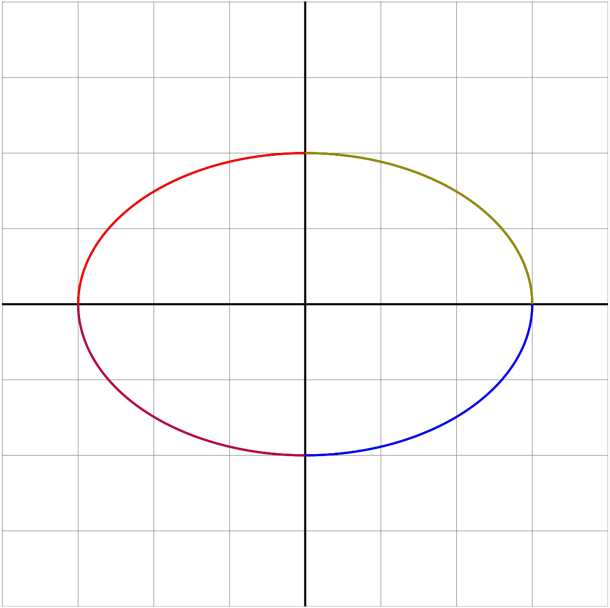


$$x^2 = y^3$$

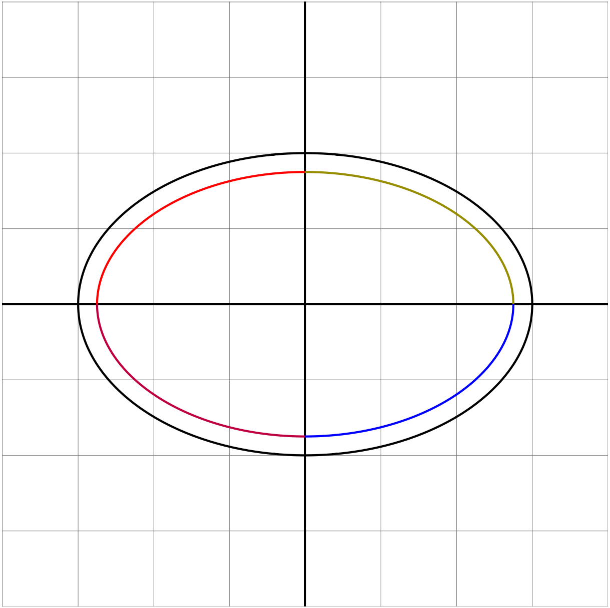
Wavefronts



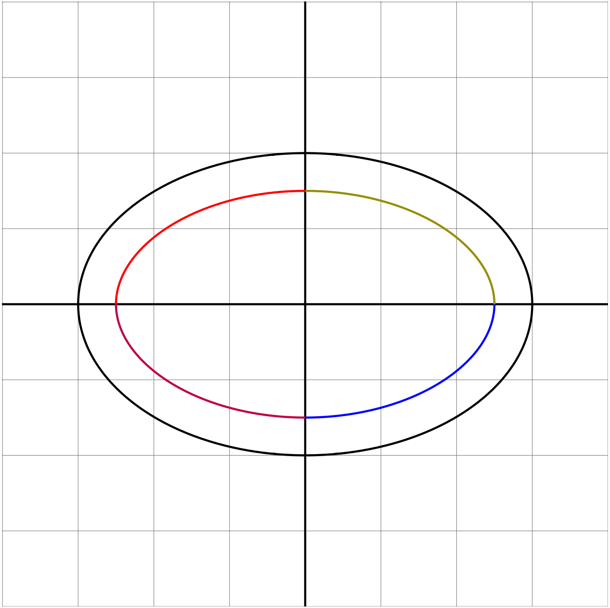
Wavefronts



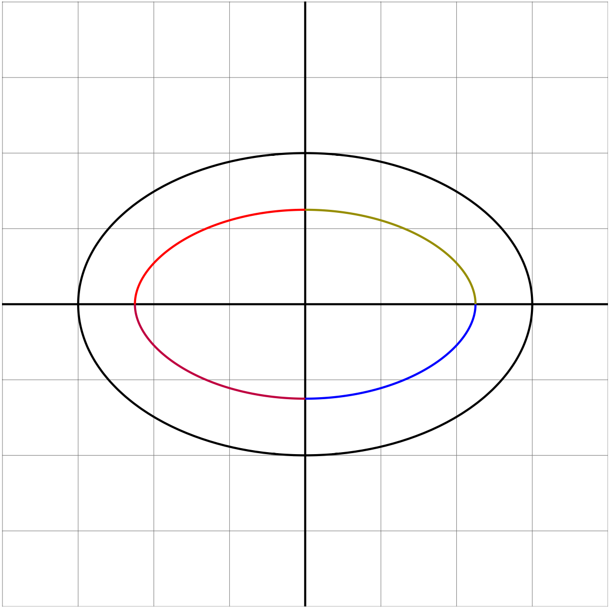
Wavefronts



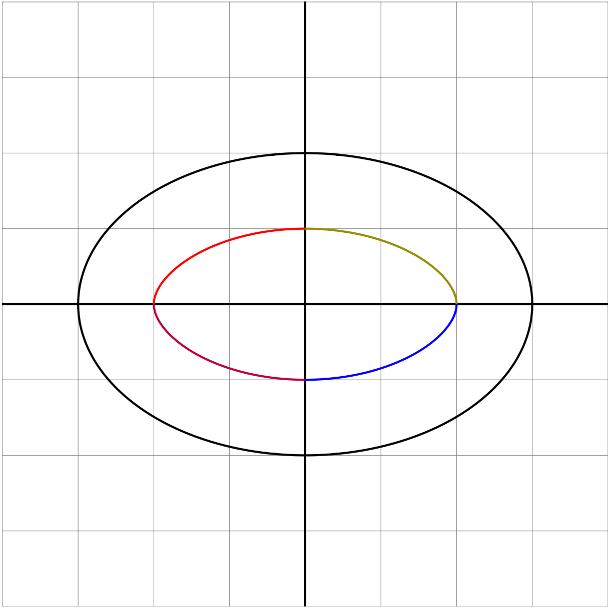
Wavefronts



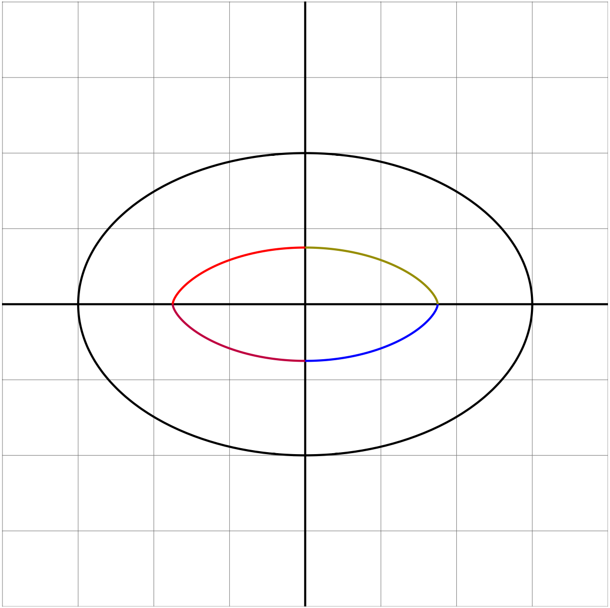
Wavefronts



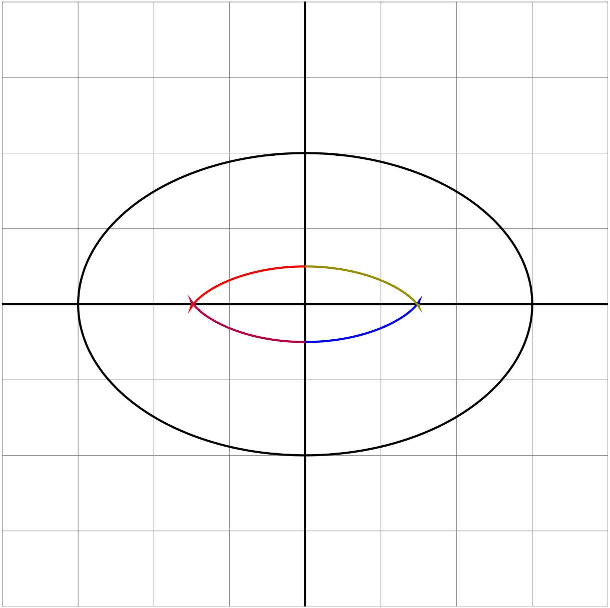
Wavefronts



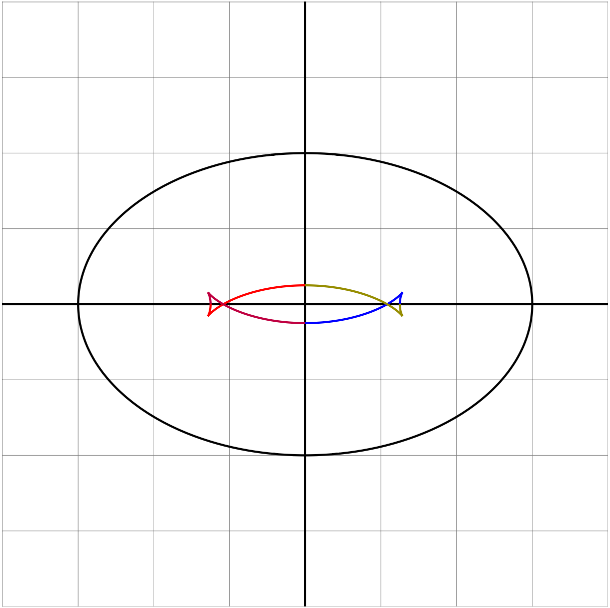
Wavefronts



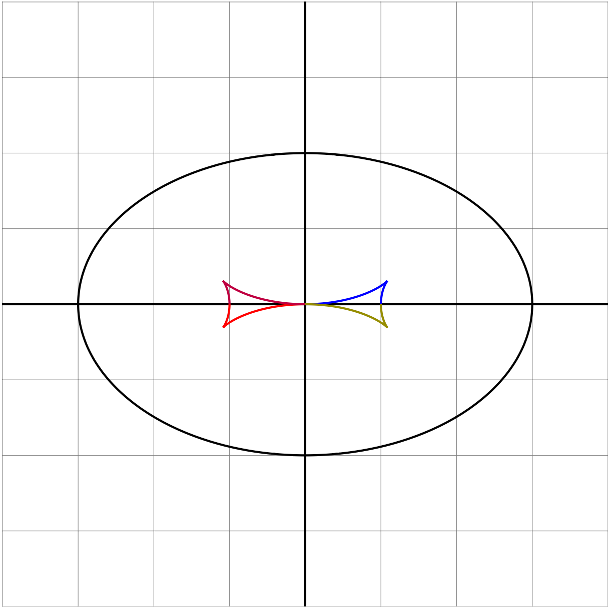
Wavefronts



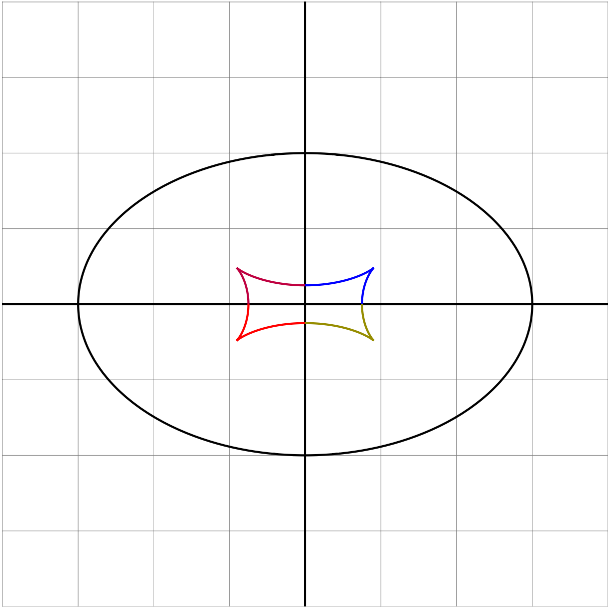
Wavefronts



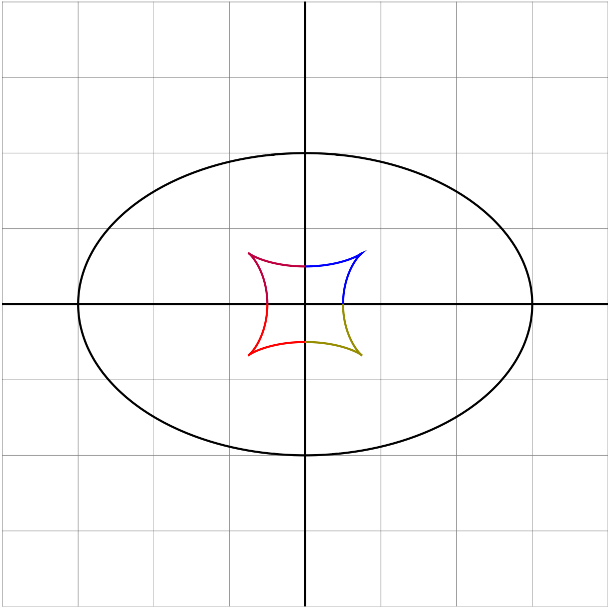
Wavefronts



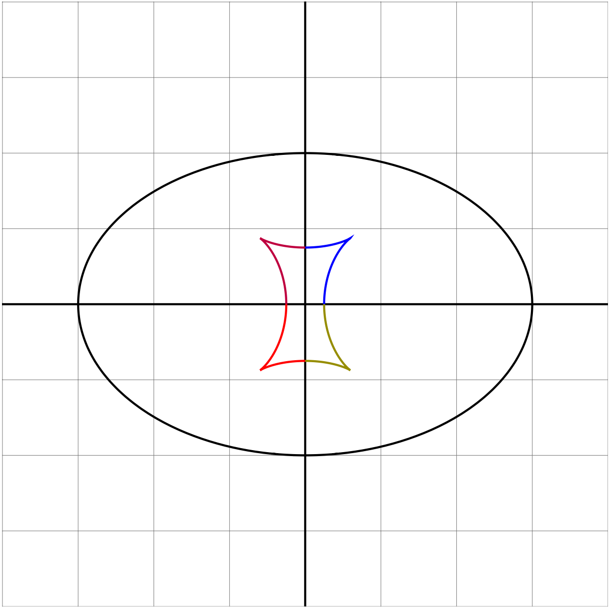
Wavefronts



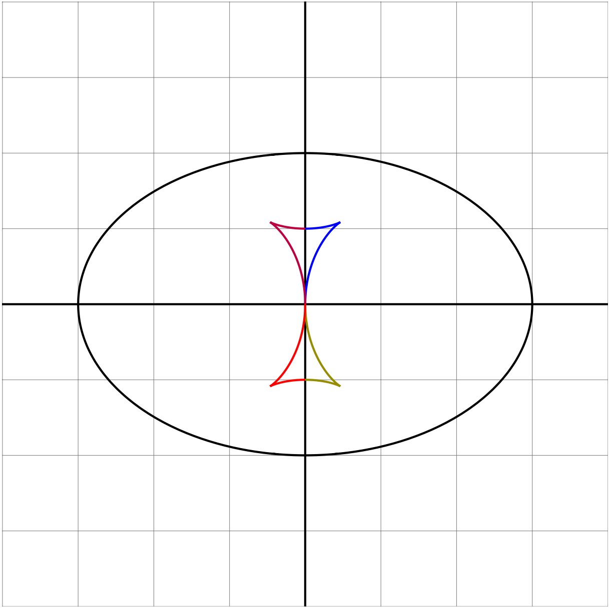
Wavefronts



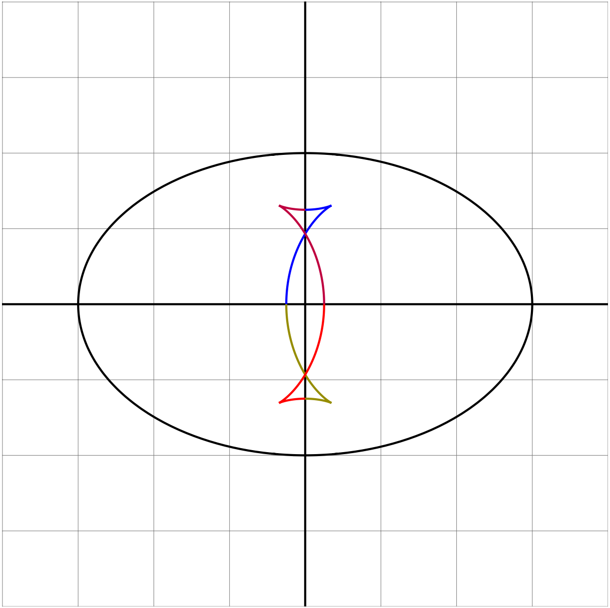
Wavefronts



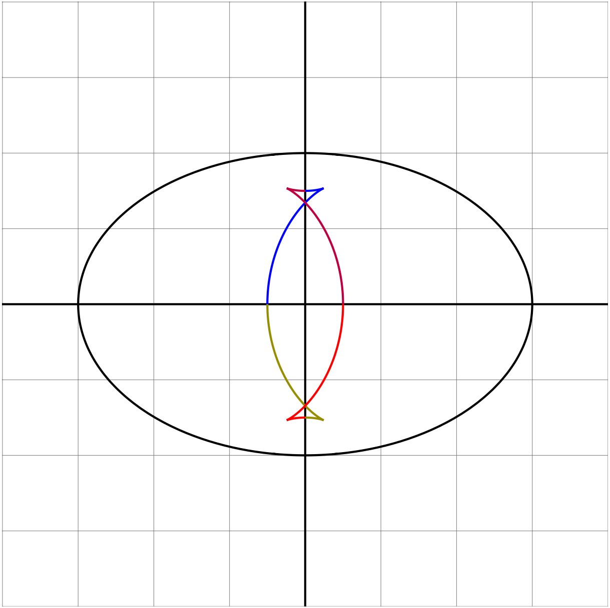
Wavefronts



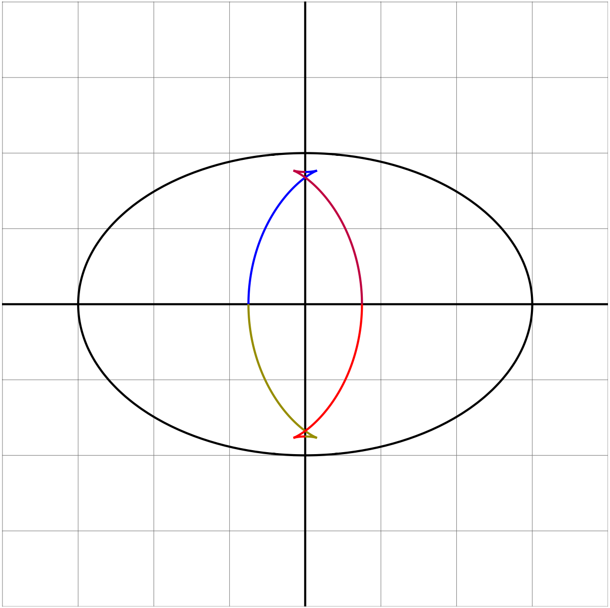
Wavefronts



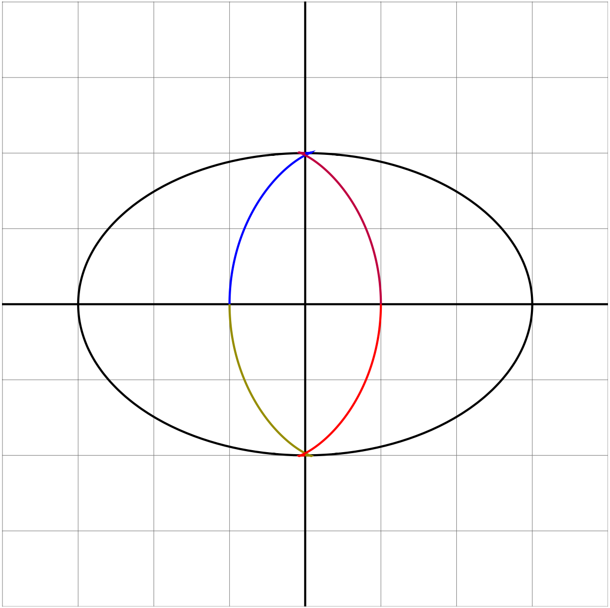
Wavefronts



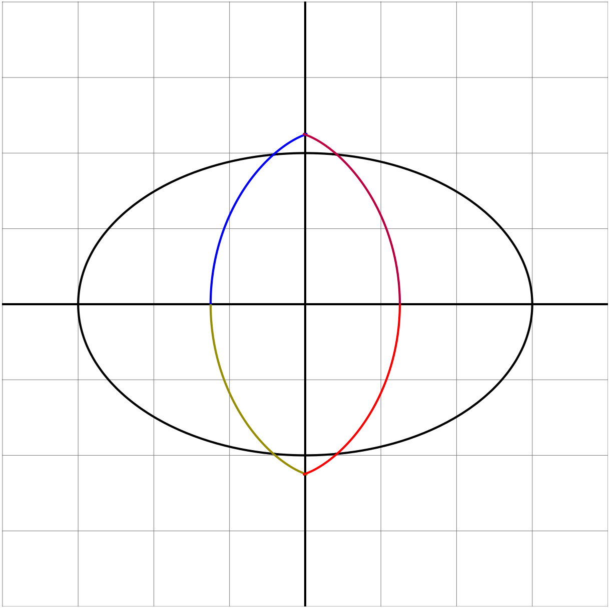
Wavefronts



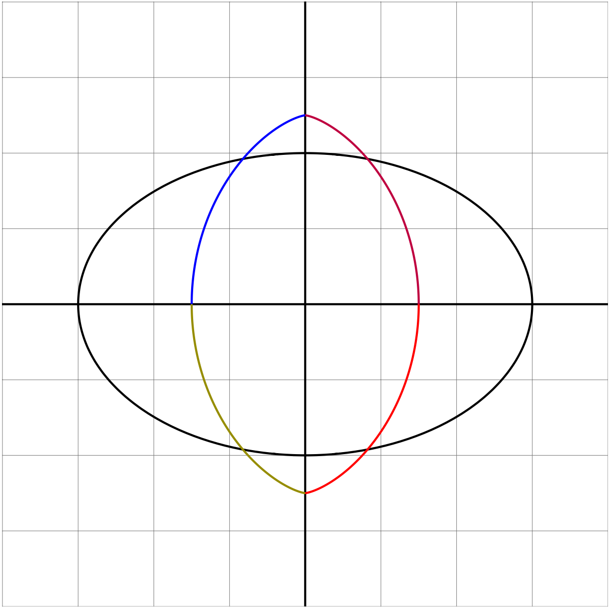
Wavefronts



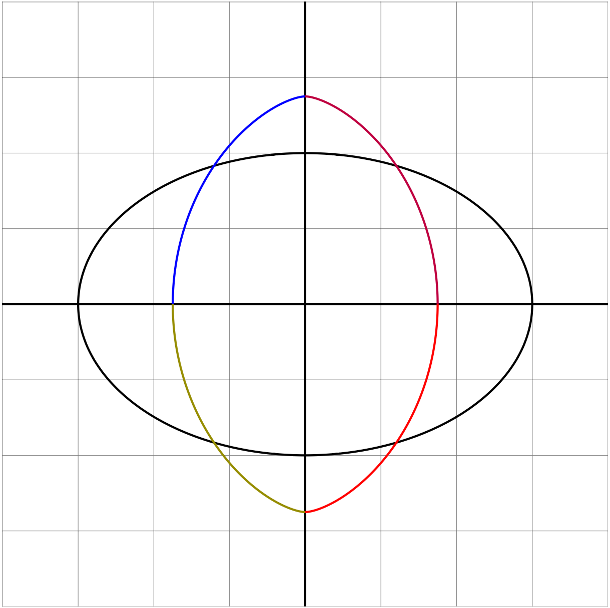
Wavefronts



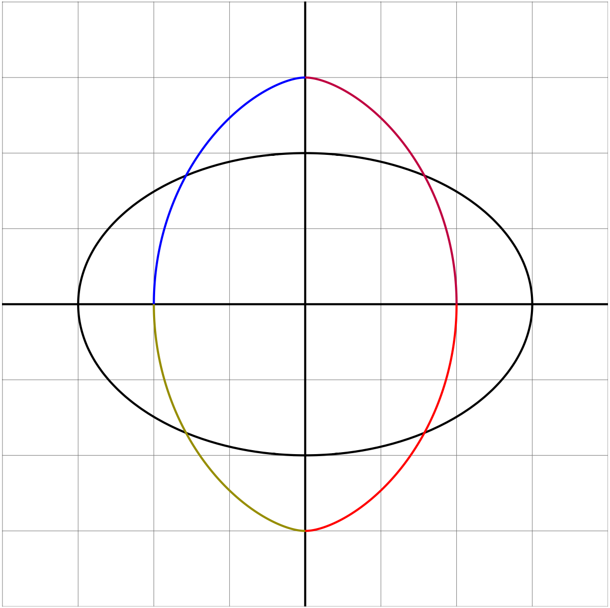
Wavefronts



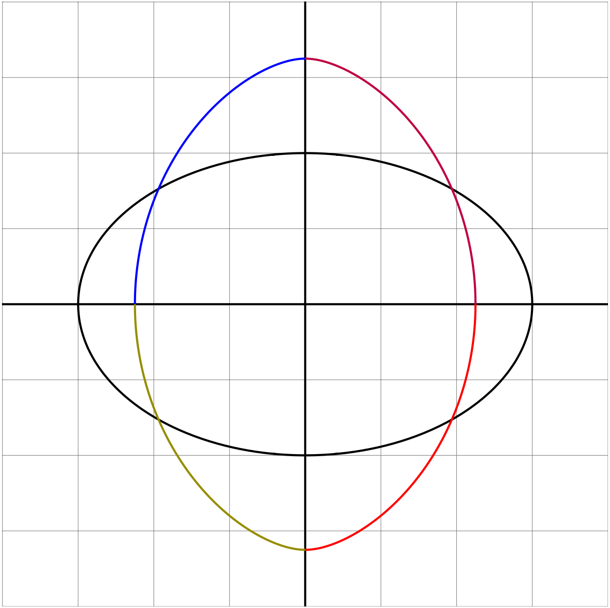
Wavefronts



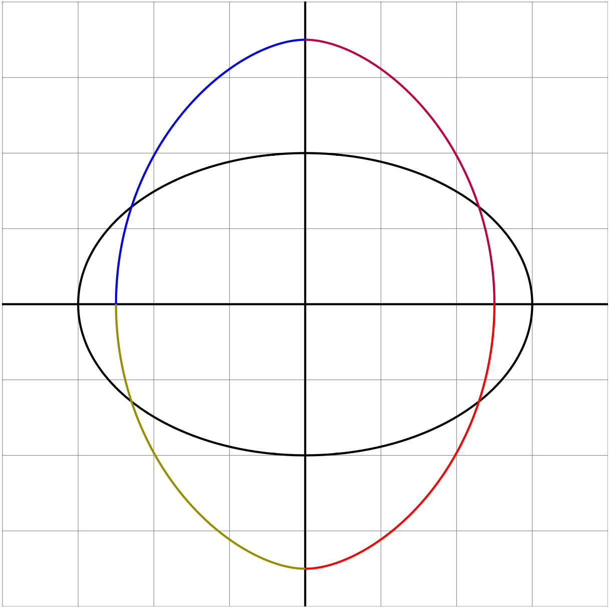
Wavefronts



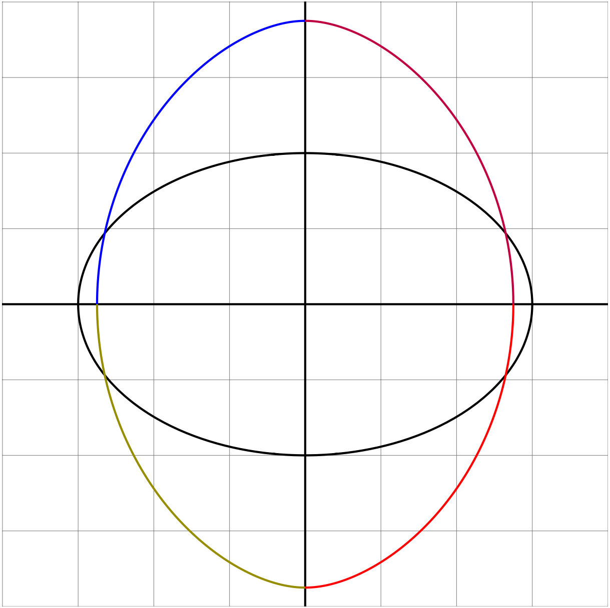
Wavefronts

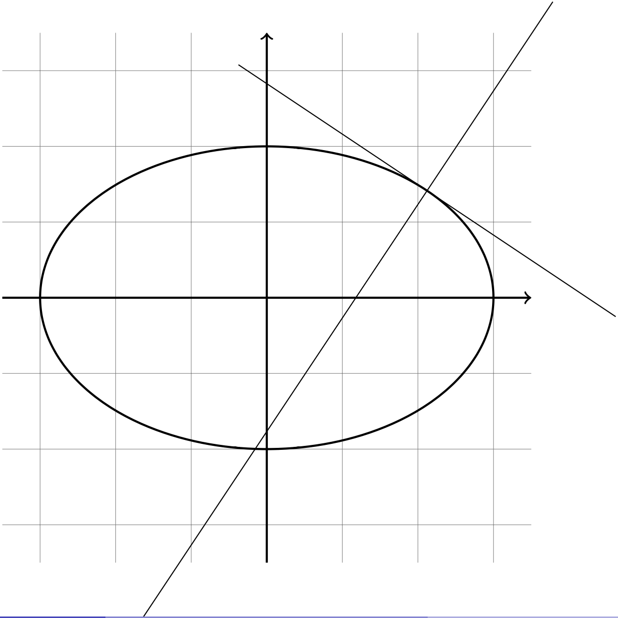


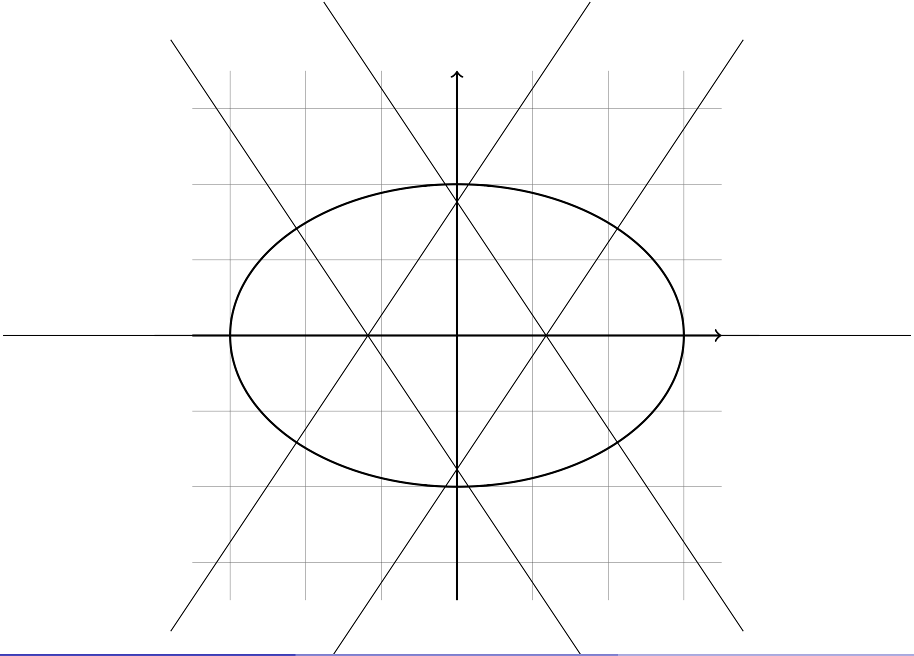
Wavefronts

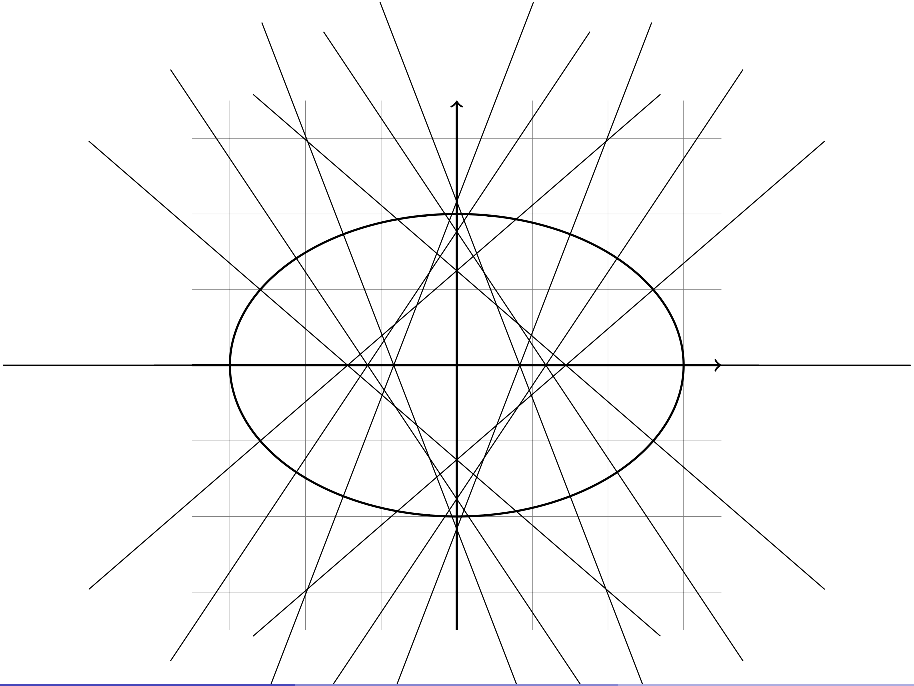


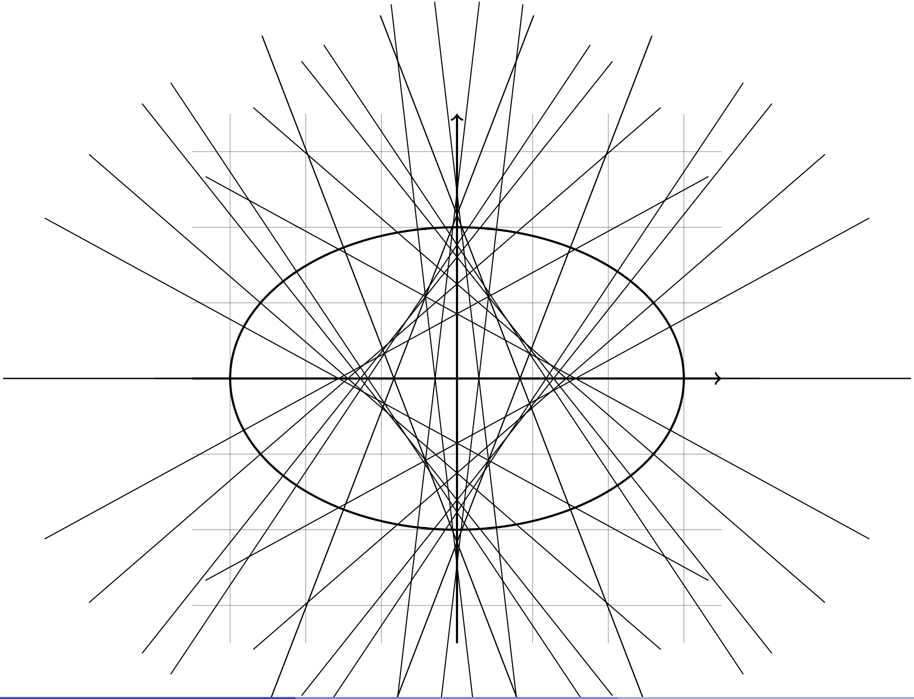
Wavefronts

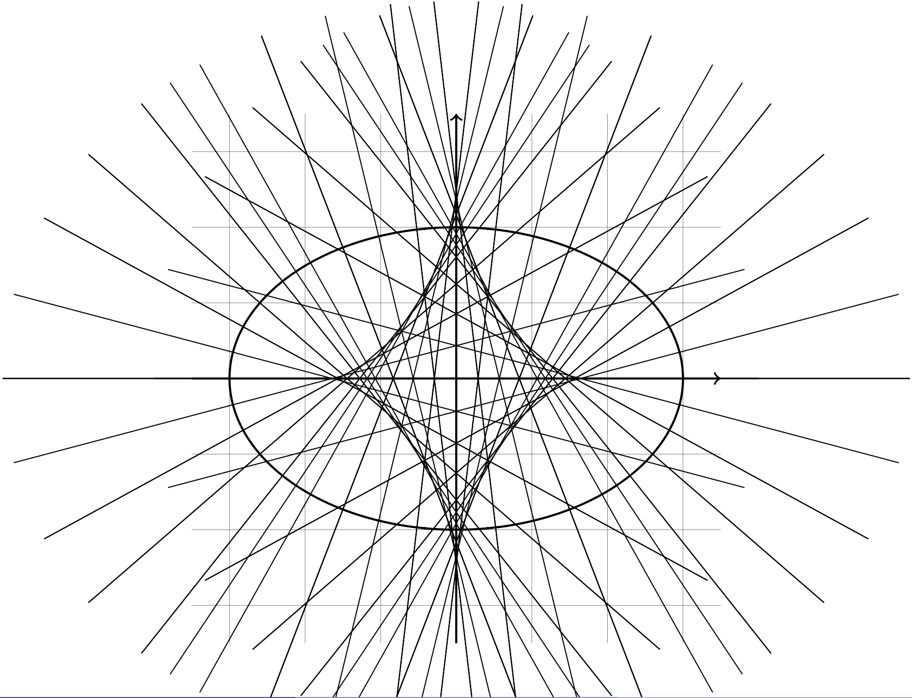


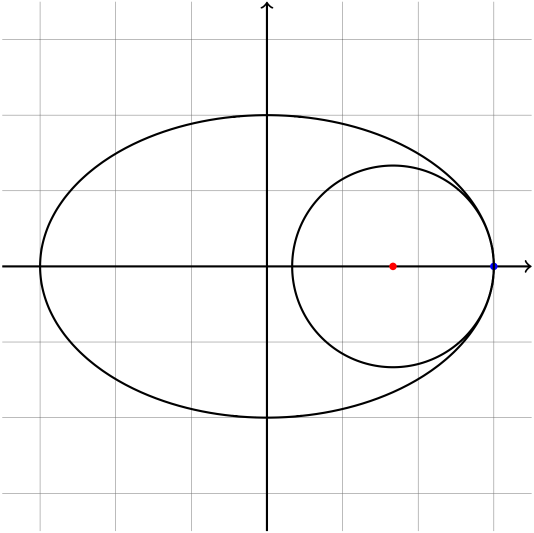


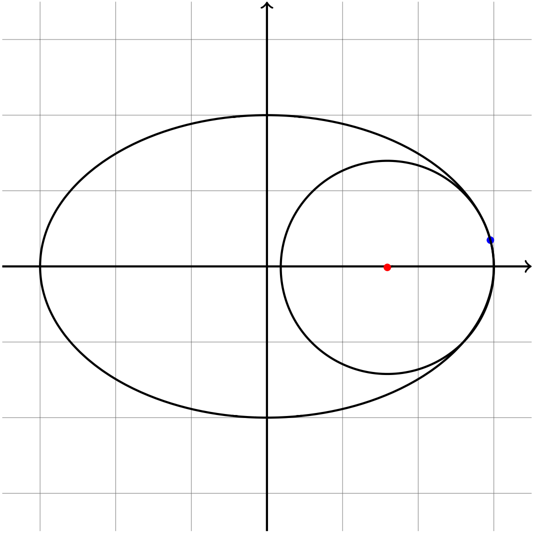


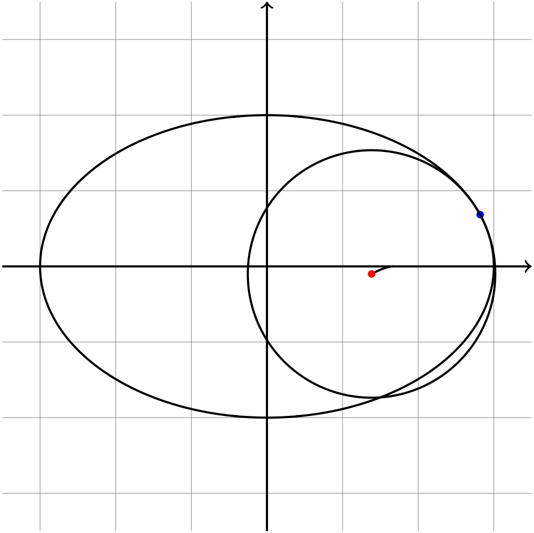


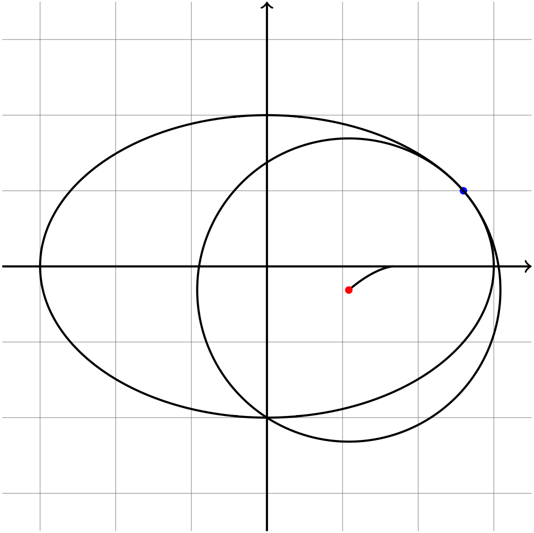


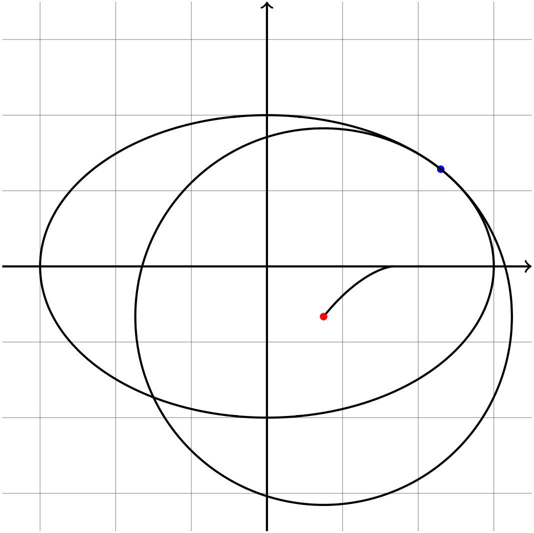


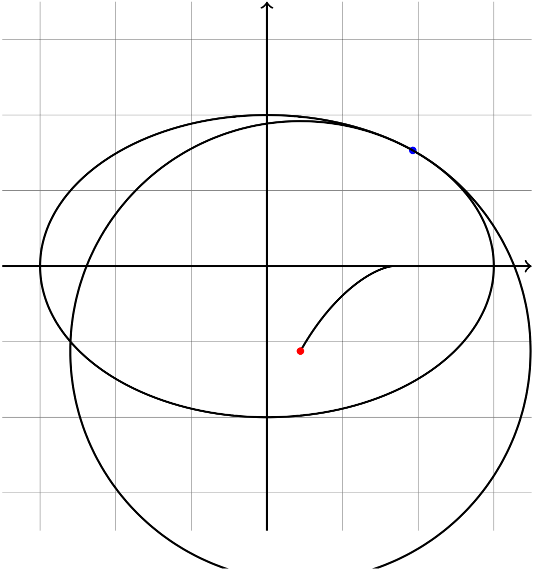


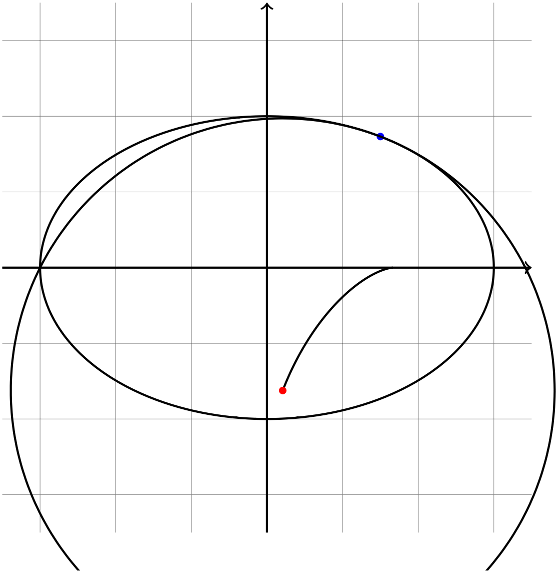


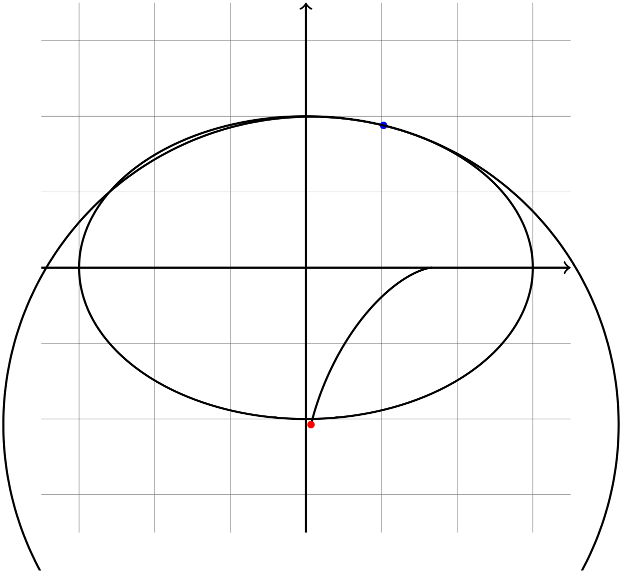


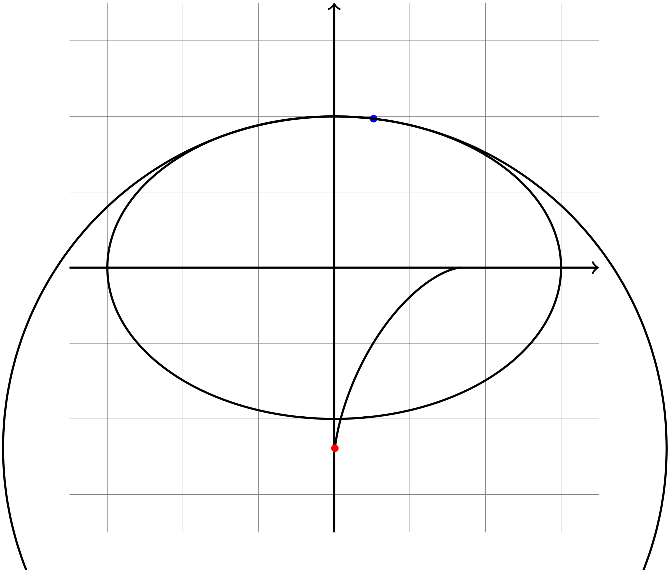


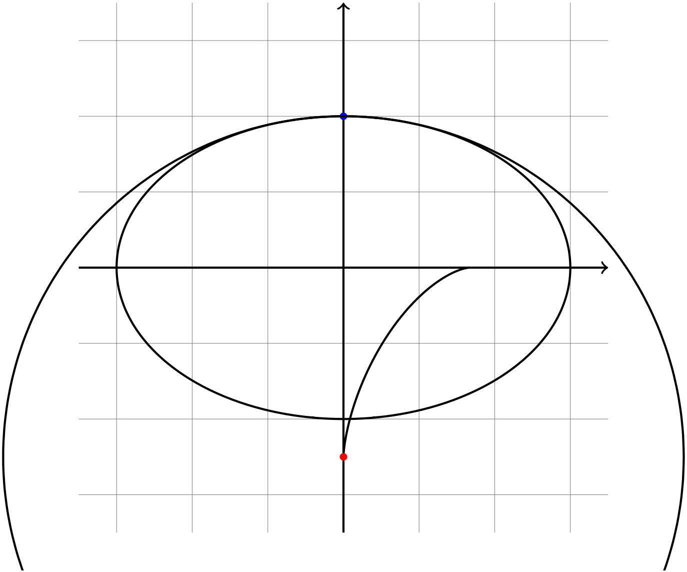


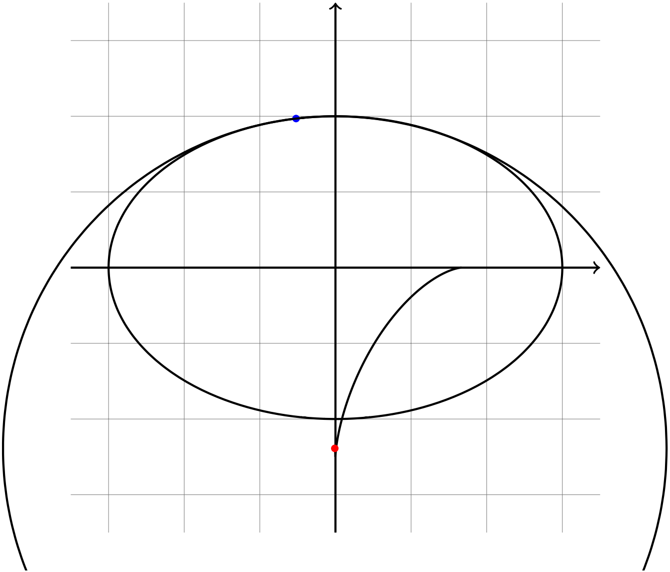


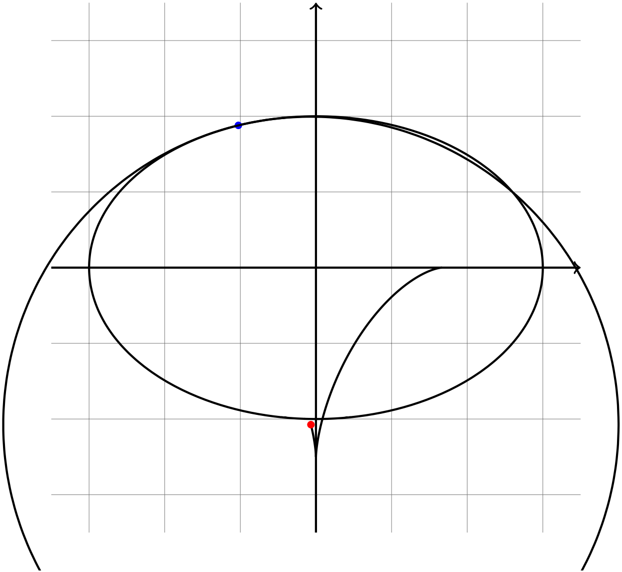


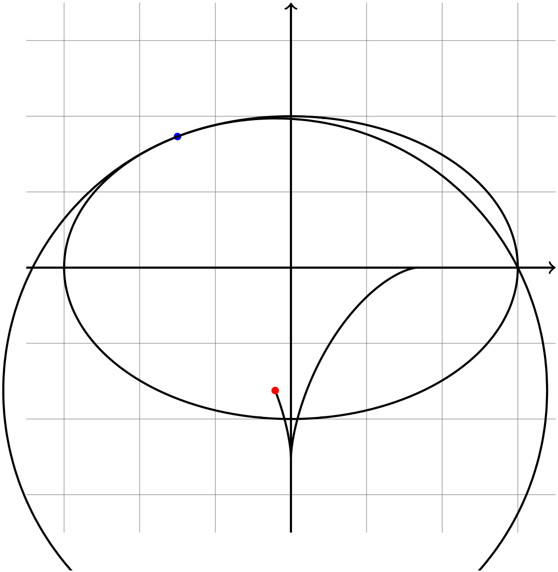


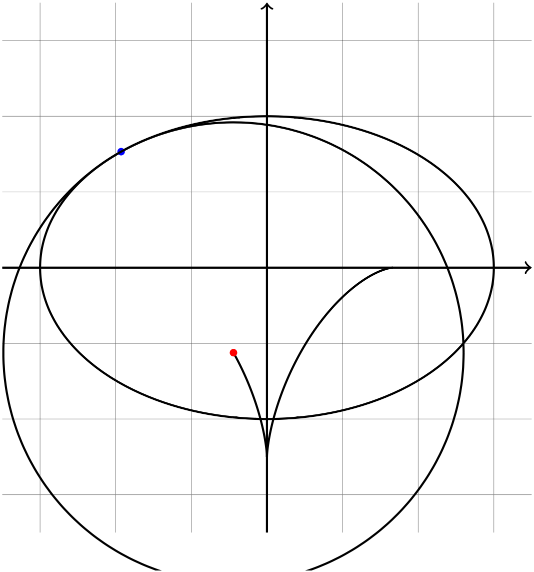


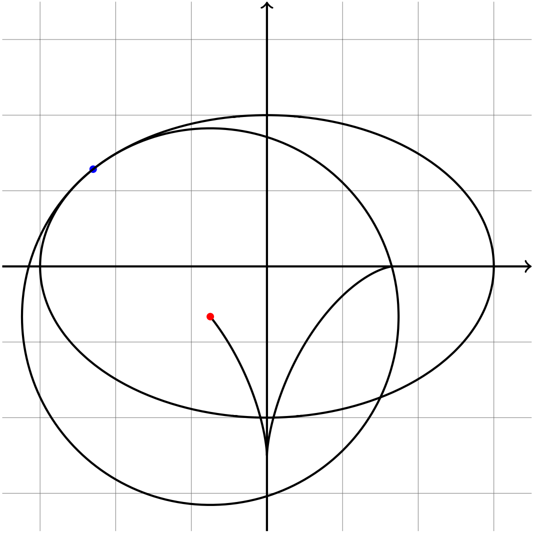


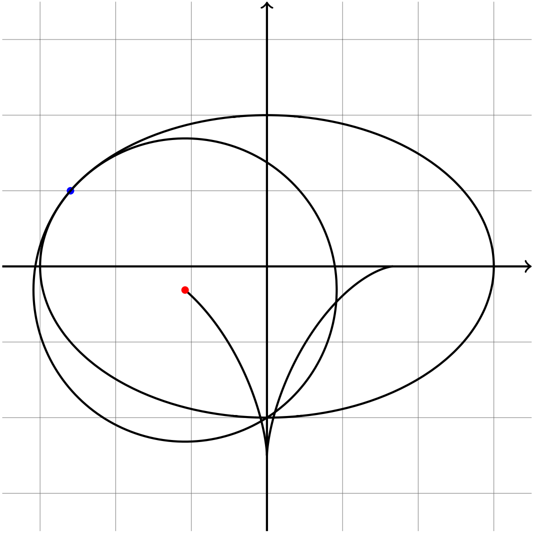


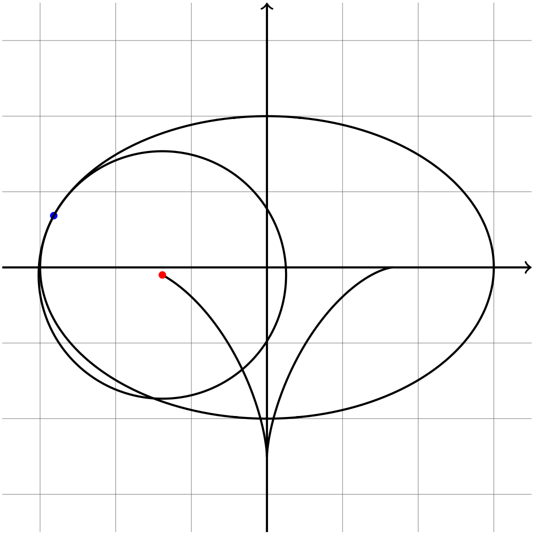


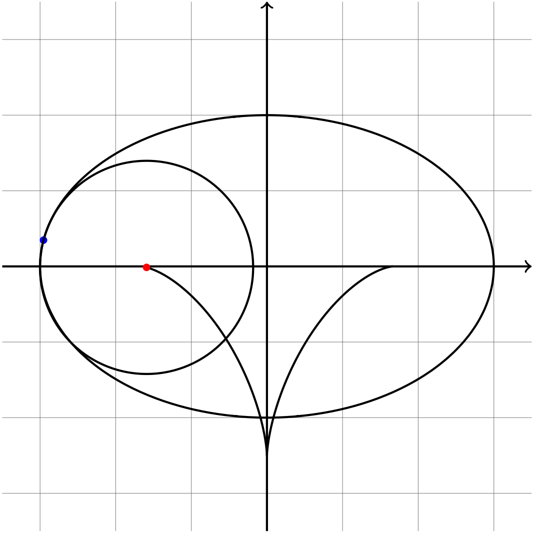


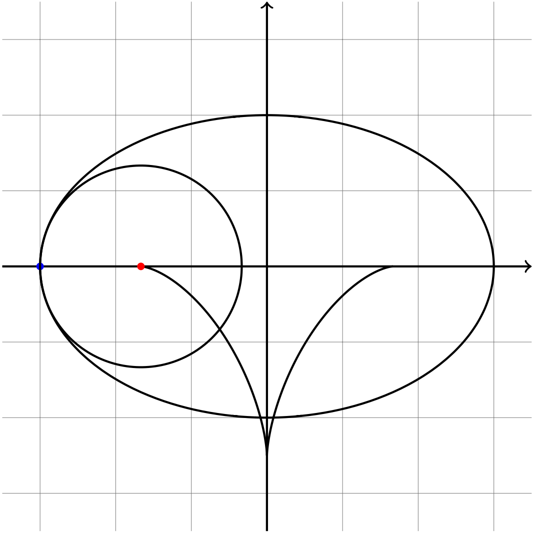


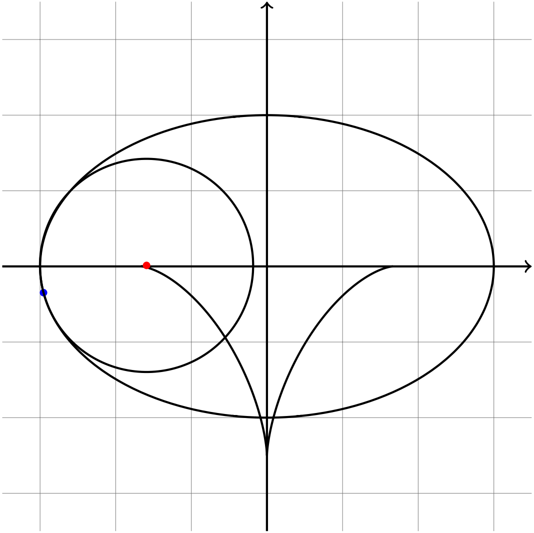


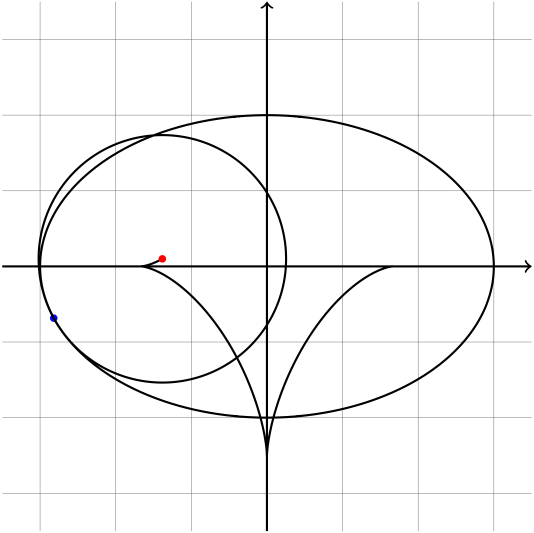


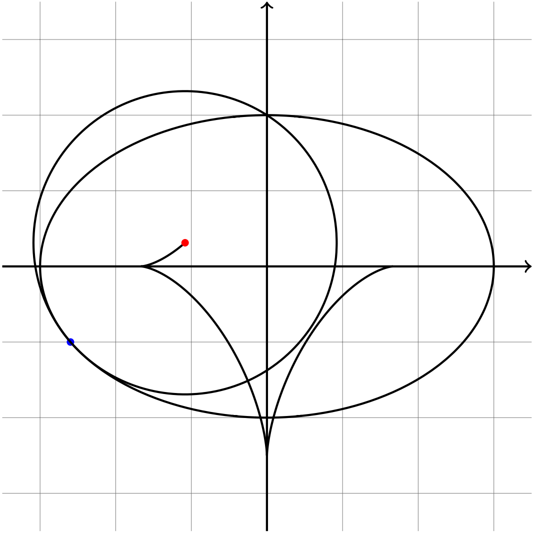


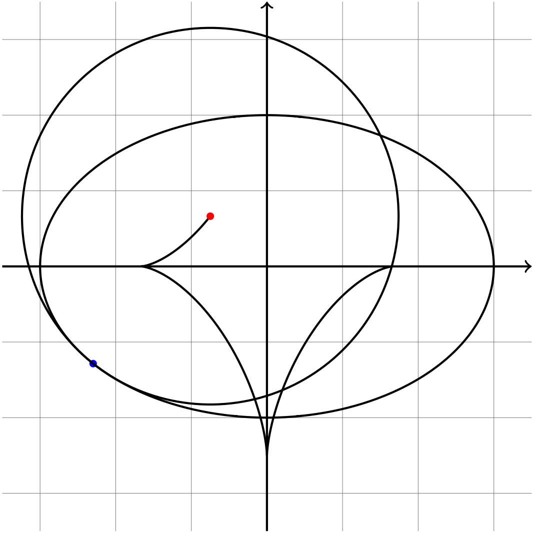


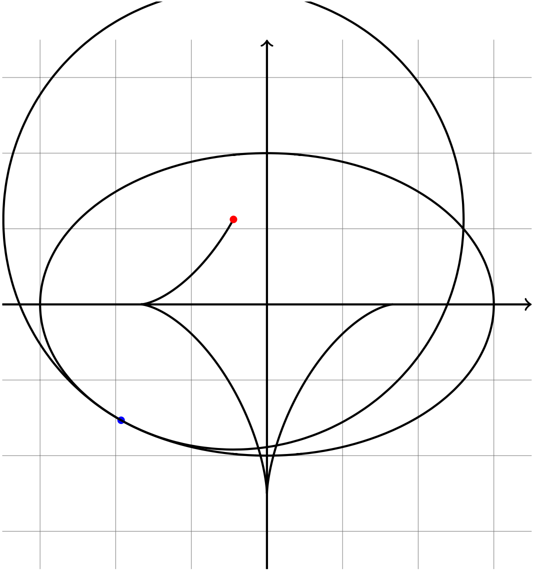


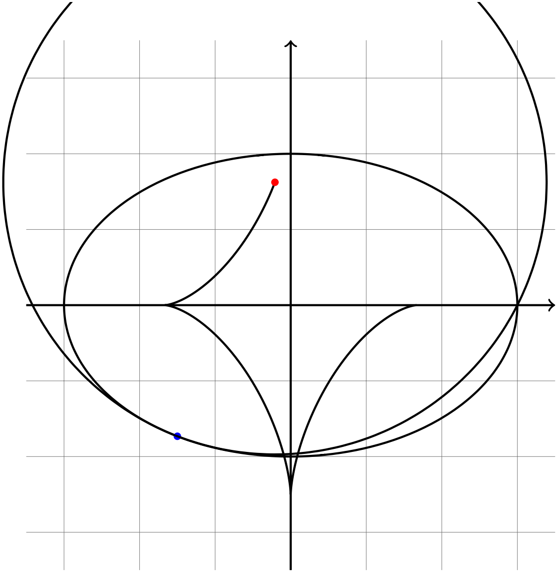


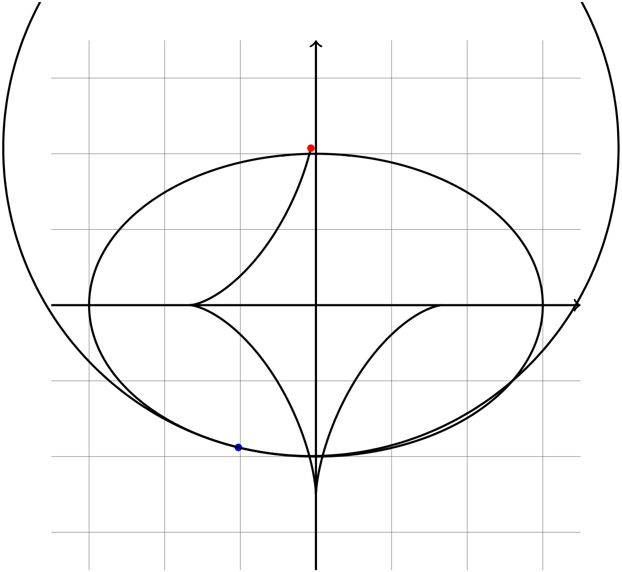


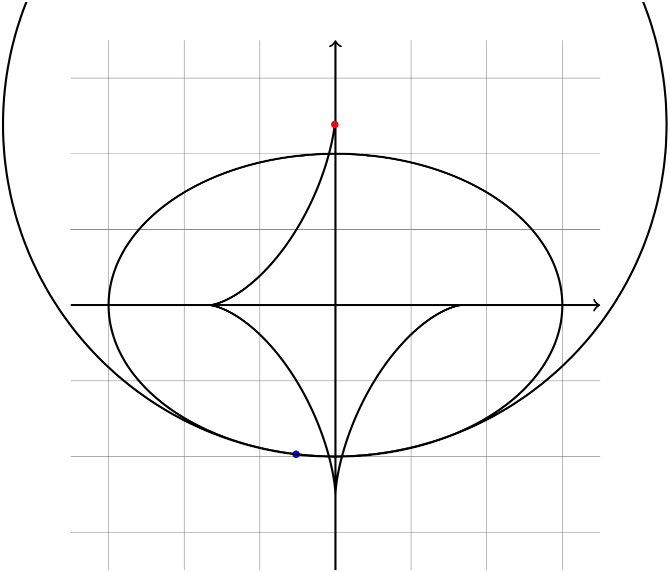


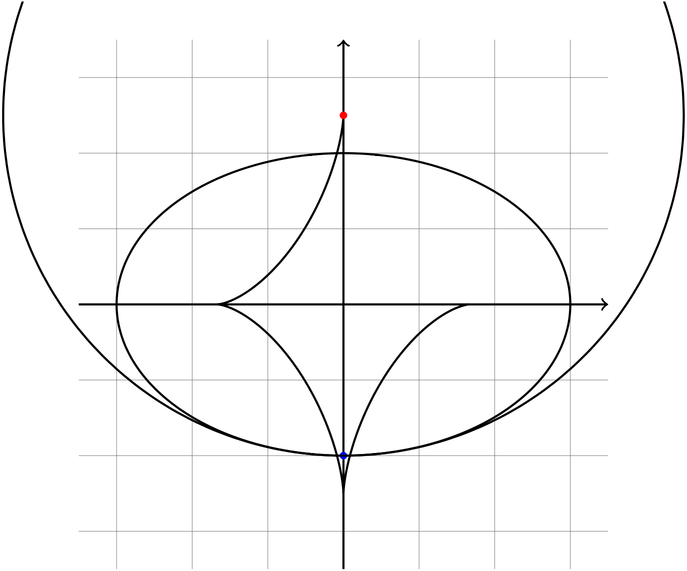


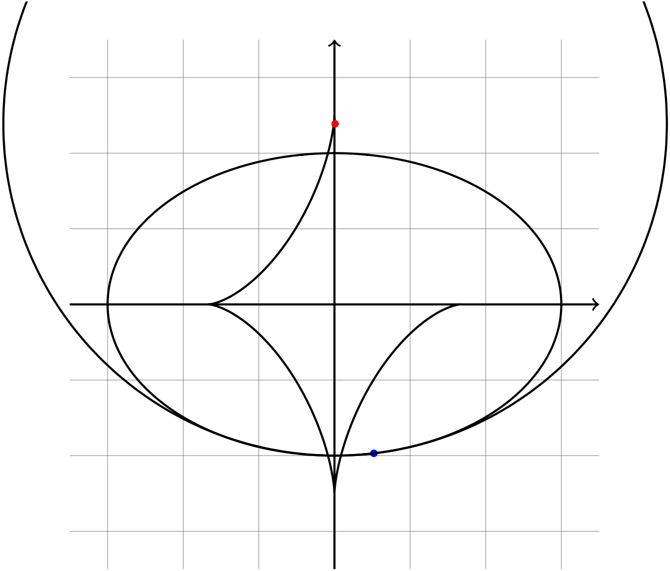


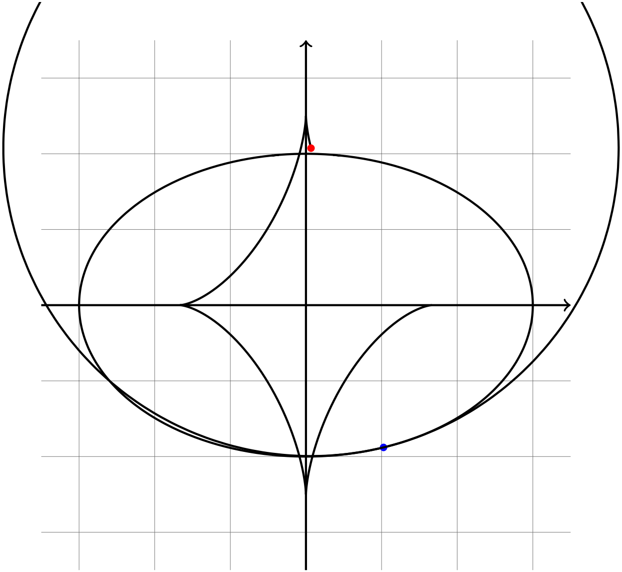


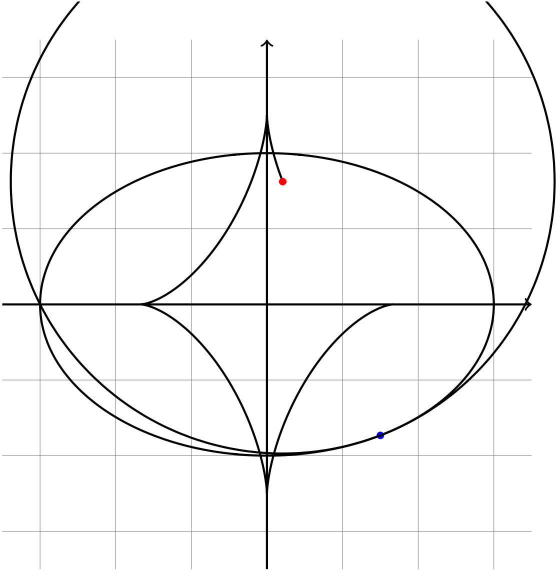


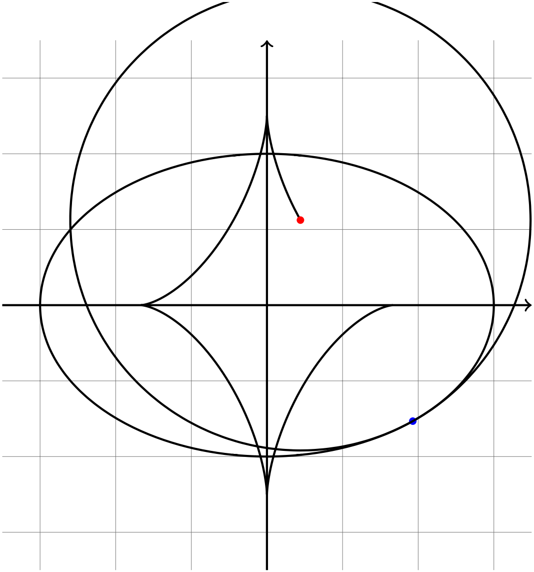


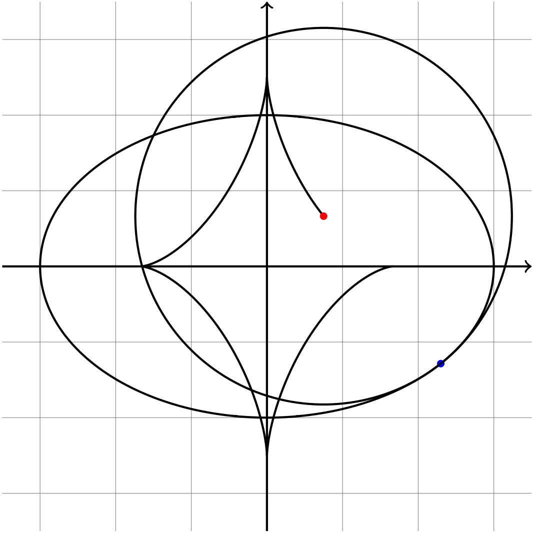


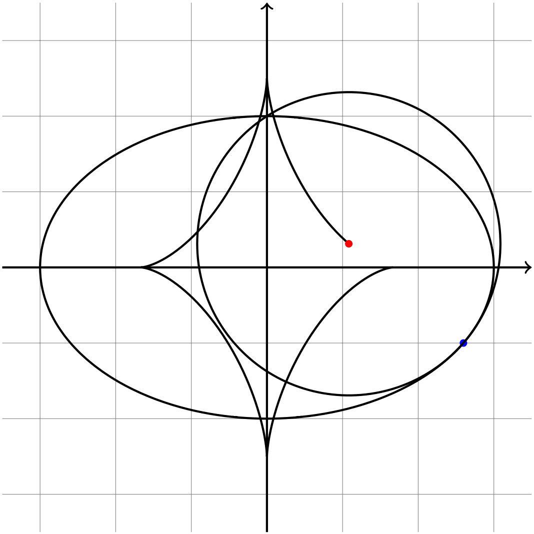


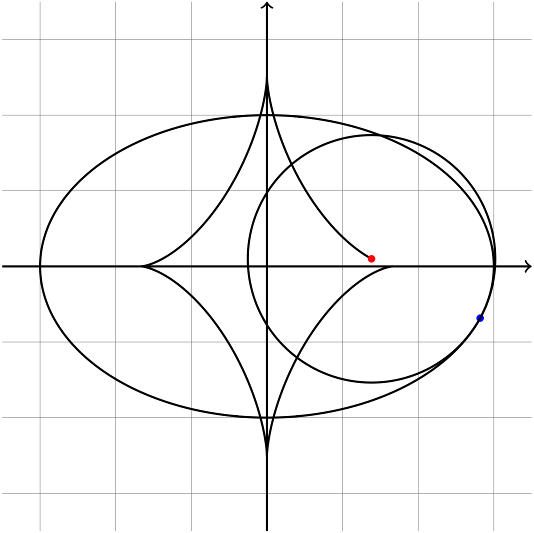


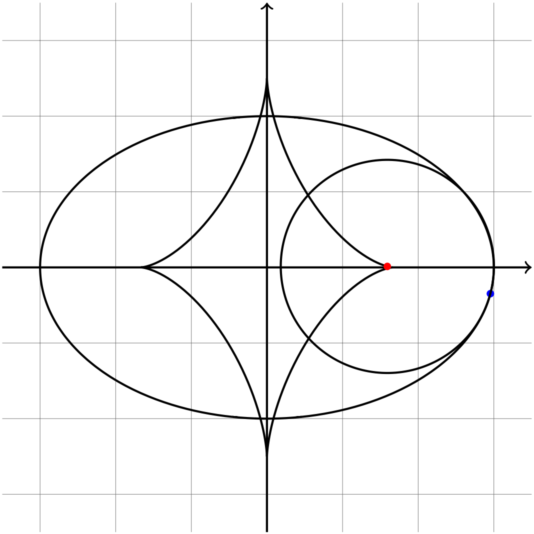


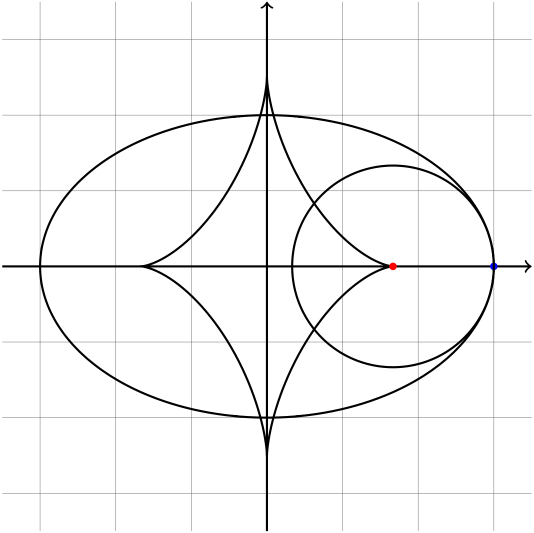












- In Chapter 10 of The Omnibus, it is proved that the envelope of the normal lines is exactly the curve traced out by the centers of curvature.

- In Chapter 10 of The Omnibus, it is proved that the envelope of the normal lines is exactly the curve traced out by the centers of curvature.
- It is also noted that the cusps appearing in the evolute is not a coincidence

- In Chapter 10 of The Omnibus, it is proved that the envelope of the normal lines is exactly the curve traced out by the centers of curvature.
- It is also noted that the cusps appearing in the evolute is not a coincidence

Theorem (Four-vertex theorem)

The curvature function of a simple, closed, smooth plane curve has at least four local extrema. Specifically, it has at least two local minima and at least two local maxima.

- In Chapter 10 of The Omnibus, it is proved that the envelope of the normal lines is exactly the curve traced out by the centers of curvature.
- It is also noted that the cusps appearing in the evolute is not a coincidence

Theorem (Four-vertex theorem)

The curvature function of a simple, closed, smooth plane curve has at least four local extrema. Specifically, it has at least two local minima and at least two local maxima.

Except for a few degenerate cases (such as circles), the extrema correspond to cusps on the evolute. For more, see Chapter 10 of The Omnibus.

Some preliminaries

We will consider algebraic curves, i.e., curves of the form

$$F = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\},$$

where $f \in \mathbb{R}[x, y]$.

Some preliminaries

We will consider algebraic curves, i.e., curves of the form

$$F = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\},$$

where $f \in \mathbb{R}[x, y]$.
Generally,

$$f(x, y) = a + b_0 x + b_1 y + c_0 x^2 + c_1 xy + c_2 x^2 + \dots$$

Some preliminaries

We will consider algebraic curves, i.e., curves of the form

$$F = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\},$$

where $f \in \mathbb{R}[x, y]$.
Generally,

$$f(x, y) = a + b_0 x + b_1 y + c_0 x^2 + c_1 xy + c_2 x^2 + \dots$$

Assume that the origin is a point on the curve, i.e., $f(0, 0) = 0$, so that $a = 0$.

Some preliminaries

We will consider algebraic curves, i.e., curves of the form

$$F = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\},$$

where $f \in \mathbb{R}[x, y]$.
Generally,

$$f(x, y) = a + b_0 x + b_1 y + c_0 x^2 + c_1 xy + c_2 x^2 + \dots$$

Assume that the origin is a point on the curve, i.e., $f(0, 0) = 0$, so that $a = 0$. Then, for there to be a cusp at the origin, it is necessary, but not sufficient, for the two partial derivatives to be 0, i.e.,

$$\frac{\partial f}{\partial x}(0, 0) = \frac{\partial f}{\partial y}(0, 0) = 0.$$

Some preliminaries

We will consider algebraic curves, i.e., curves of the form

$$F = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\},$$

where $f \in \mathbb{R}[x, y]$.
Generally,

$$f(x, y) = a + b_0 x + b_1 y + c_0 x^2 + c_1 xy + c_2 x^2 + \dots$$

Assume that the origin is a point on the curve, i.e., $f(0, 0) = 0$, so that $a = 0$. Then, for there to be a cusp at the origin, it is necessary, but not sufficient, for the two partial derivatives to be 0, i.e.,

$$\frac{\partial f}{\partial x}(0, 0) = \frac{\partial f}{\partial y}(0, 0) = 0.$$

Thus, if there is a cusp at the origin, we have $a = b_0 = b_1 = 0$.

The "definition" of a cusp

Let $f(x, y)$ be a polynomial of the following form:

$$f(x, y) = c_0x^2 + c_1xy + c_2y^2 + d_0x^3 + d_1x^2y + d_2xy^2 + d_3y^3 + \dots$$

The "definition" of a cusp

Let $f(x, y)$ be a polynomial of the following form:

$$f(x, y) = c_0x^2 + c_1xy + c_2y^2 + d_0x^3 + d_1x^2y + d_2xy^2 + d_3y^3 + \dots$$

Some sources *define* a cusp in terms of the coefficients.

The "definition" of a cusp

Let $f(x, y)$ be a polynomial of the following form:

$$f(x, y) = c_0x^2 + c_1xy + c_2y^2 + d_0x^3 + d_1x^2y + d_2xy^2 + d_3y^3 + \dots$$

Some sources *define* a cusp in terms of the coefficients.

Definition

Let F be an algebraic curve defined by a polynomial f as above. If

- at least one of c_0, c_1, c_2 is non-zero,
- at least one of d_0, d_1, d_2, d_3 is non-zero, and
- there is one real solution in m of multiplicity 2 to the polynomial

$$c_0 + c_1m + c_2m^2,$$

then somehow there is a cusp at the origin.

Consider the semicubic parabola, given by the roots of

$$f(x, y) = y^2 - x^3.$$

Consider the semicubic parabola, given by the roots of

$$f(x,y) = y^2 - x^3.$$

We have $a = b_0 = b_1 = c_0 = c_1 = 0$, but $c_2 = 1$ and $d_0 = -1$.

Consider the semicubic parabola, given by the roots of

$$f(x, y) = y^2 - x^3.$$

We have $a = b_0 = b_1 = c_0 = c_1 = 0$, but $c_2 = 1$ and $d_0 = -1$. The polynomial

$$c_0 + c_1 m + c_2 m^2 = m^2$$

Consider the semicubic parabola, given by the roots of

$$f(x, y) = y^2 - x^3.$$

We have $a = b_0 = b_1 = c_0 = c_1 = 0$, but $c_2 = 1$ and $d_0 = -1$. The polynomial

$$c_0 + c_1 m + c_2 m^2 = m^2$$

has a double root $m = 0$, so there is a cusp at the origin.

Consider the semicubic parabola, given by the roots of

$$f(x, y) = y^2 - x^3.$$

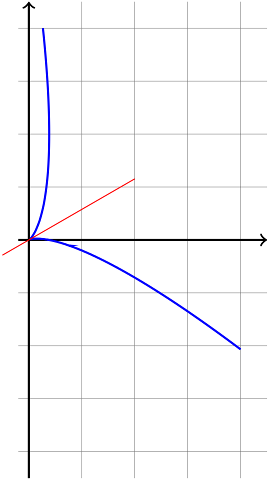
We have $a = b_0 = b_1 = c_0 = c_1 = 0$, but $c_2 = 1$ and $d_0 = -1$. The polynomial

$$c_0 + c_1 m + c_2 m^2 = m^2$$

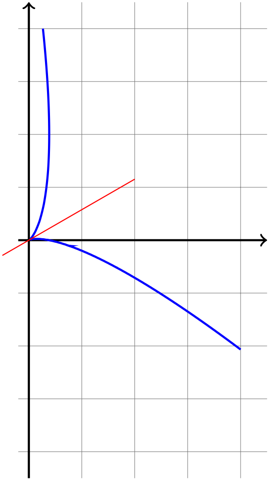
has a double root $m = 0$, so there is a cusp at the origin.
You may also observe from earlier that the slope of both tangents in the diagram from earlier approaches 0 as we approach the cusp.

Let's rotate the semicubical parabola 30 degrees anticlockwise and see what happens.

Let's rotate the semicubical parabola 30 degrees anticlockwise and see what happens.



Let's rotate the semicubical parabola 30 degrees anticlockwise and see what happens.



The slope of the tangents now approaches a value of $\tan 30^\circ = \frac{\sqrt{3}}{3}$.

To rotate the curve about an angle of θ, use the following change of variables:

$$(x, y) \mapsto (x\cos\theta + y\sin\theta, y\cos\theta - x\sin\theta)$$

To rotate the curve about an angle of θ, use the following change of variables:

$$(x, y) \mapsto (x\cos\theta + y\sin\theta, y\cos\theta - x\sin\theta)$$

Changing the variables in $x^3 - y^2$ gives

$$(x\cos\theta + y\sin\theta)^3 - (y\cos\theta - x\sin\theta)^2$$

To rotate the curve about an angle of θ, use the following change of variables:

$$(x, y) \mapsto (x\cos\theta + y\sin\theta, y\cos\theta - x\sin\theta)$$

Changing the variables in $x^3 - y^2$ gives

$$(x\cos\theta + y\sin\theta)^3 - (y\cos\theta - x\sin\theta)^2$$

In that case, expanding gives $a = b_0 = b_1 = 0$, and at least one of the d_i's is non-zero.

To rotate the curve about an angle of θ, use the following change of variables:

$$(x, y) \mapsto (x\cos\theta + y\sin\theta, y\cos\theta - x\sin\theta)$$

Changing the variables in $x^3 - y^2$ gives

$$(x\cos\theta + y\sin\theta)^3 - (y\cos\theta - x\sin\theta)^2$$

In that case, expanding gives $a = b_0 = b_1 = 0$, and at least one of the d_i's is non-zero.
Moreover, $c_0 = \sin^2\theta$, $c_1 = -2\sin\theta\cos\theta$, $c_2 = \cos^2\theta$.

To rotate the curve about an angle of θ, use the following change of variables:

$$(x, y) \mapsto (x \cos\theta + y \sin\theta, y \cos\theta - x \sin\theta)$$

Changing the variables in $x^3 - y^2$ gives

$$(x \cos\theta + y \sin\theta)^3 - (y \cos\theta - x \sin\theta)^2$$

In that case, expanding gives $a = b_0 = b_1 = 0$, and at least one of the d_i's is non-zero.
Moreover, $c_0 = \sin^2\theta$, $c_1 = -2\sin\theta\cos\theta$, $c_2 = \cos^2\theta$.
So the polynomial of interest is

$$m^2\cos^2\theta - 2m\sin\theta\cos\theta + \sin^2\theta,$$

To rotate the curve about an angle of θ, use the following change of variables:

$$(x, y) \mapsto (x\cos\theta + y\sin\theta, y\cos\theta - x\sin\theta)$$

Changing the variables in $x^3 - y^2$ gives

$$(x\cos\theta + y\sin\theta)^3 - (y\cos\theta - x\sin\theta)^2$$

In that case, expanding gives $a = b_0 = b_1 = 0$, and at least one of the d_i's is non-zero.
Moreover, $c_0 = \sin^2\theta$, $c_1 = -2\sin\theta\cos\theta$, $c_2 = \cos^2\theta$.
So the polynomial of interest is

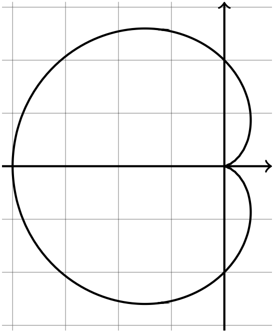
$$m^2\cos^2\theta - 2m\sin\theta\cos\theta + \sin^2\theta,$$

which has a double root $m = \tan\theta$ provided that $\cos\theta \neq 0$.

Somehow, the solution m is the “tangent” at the cusp.

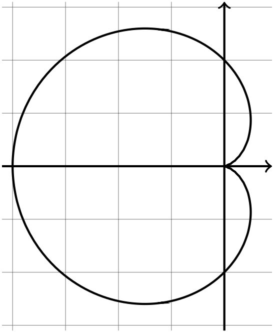
Another classic example of a cusp is in the cardioid, defined by the polar equation

$$r(\theta) = 2(1 - \cos\theta).$$



Another classic example of a cusp is in the cardioid, defined by the polar equation

$$r(\theta) = 2(1 - \cos\theta).$$



Substituting $\frac{x}{r}$ for $\cos\theta$ and $\sqrt{x^2 + y^2}$ for r results in the implicit representation

$$f(x, y) = (x^2 + y^2)^2 + 4x(x^2 + y^2) - 4y^2 = 0$$

Expanding the cardioid polynomial, we get

$$f(x,y) = -4y^2 + 4x^3 + 4xy^2 + x^4 + 2x^2y^2 + y^4$$

In this case, $a = b_0 = b_1 = c_0 = c_1 = d_2 = d_3 = 0$, but $c_2 = -4$ and $d_1 = 4$.

Expanding the cardioid polynomial, we get

$$f(x,y) = -4y^2 + 4x^3 + 4xy^2 + x^4 + 2x^2y^2 + y^4$$

In this case, $a = b_0 = b_1 = c_0 = c_1 = d_2 = d_3 = 0$, but $c_2 = -4$ and $d_1 = 4$. So the "tangent" is given by the solution to

$$c_2 m^2 = -4m^2 = 0,$$

Expanding the cardioid polynomial, we get

$$f(x,y) = -4y^2 + 4x^3 + 4xy^2 + x^4 + 2x^2y^2 + y^4$$

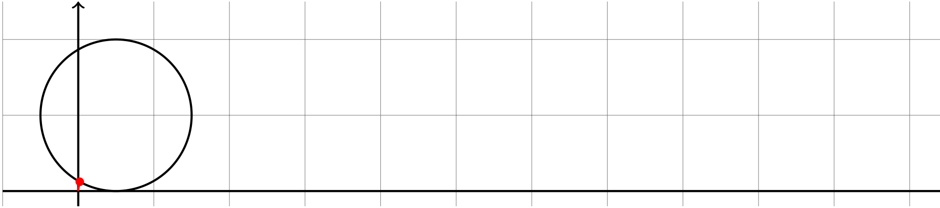
In this case, $a = b_0 = b_1 = c_0 = c_1 = d_2 = d_3 = 0$, but $c_2 = -4$ and $d_1 = 4$. So the "tangent" is given by the solution to

$$c_2 m^2 = -4m^2 = 0,$$

which clearly has a double root $m = 0$.

More cusps

More cusps



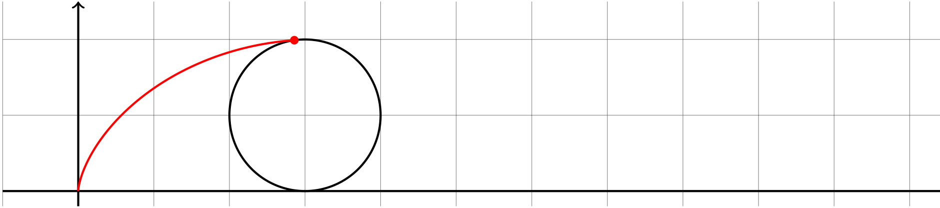
More cusps

More cusps

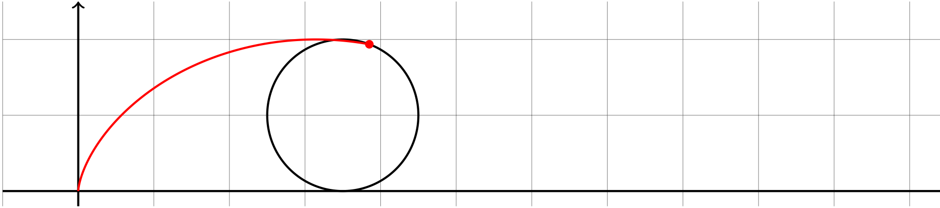
More cusps

More cusps

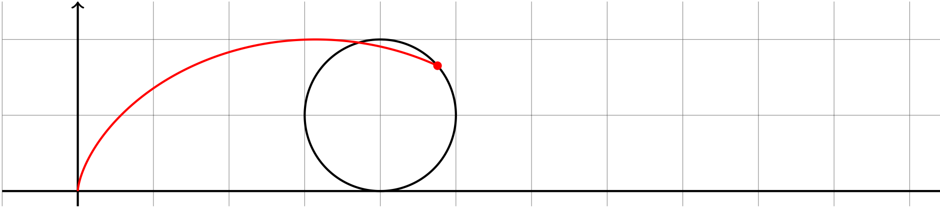
More cusps



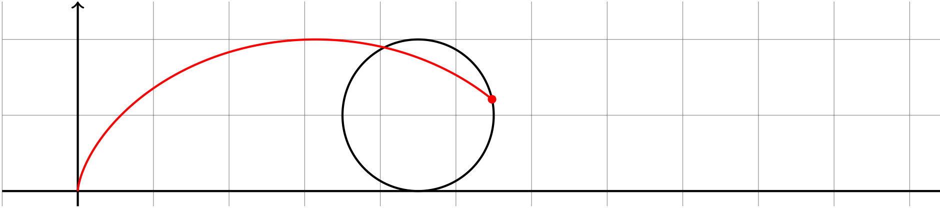
More cusps



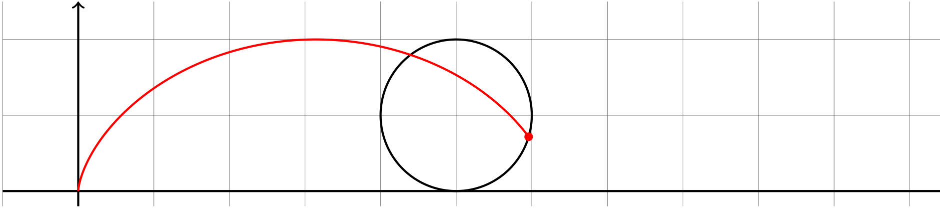
More cusps



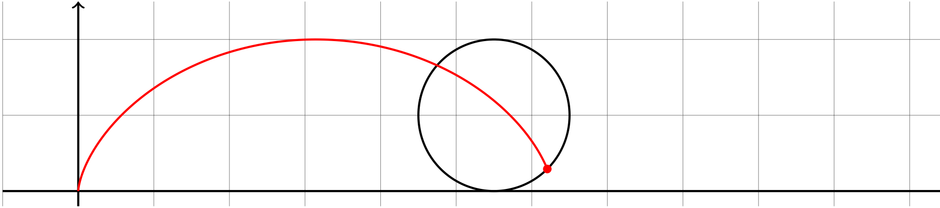
More cusps



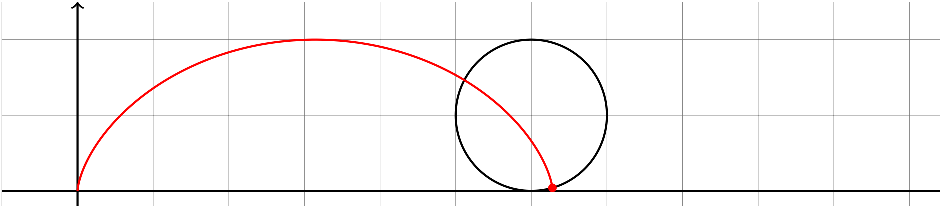
More cusps



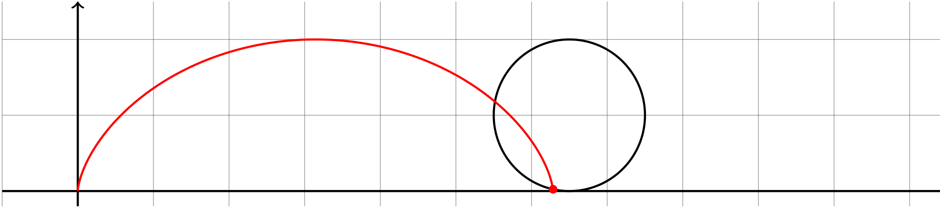
More cusps



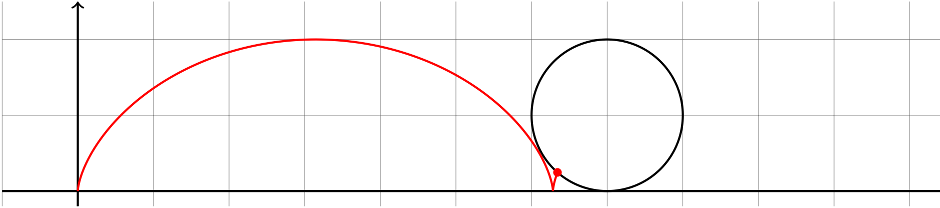
More cusps



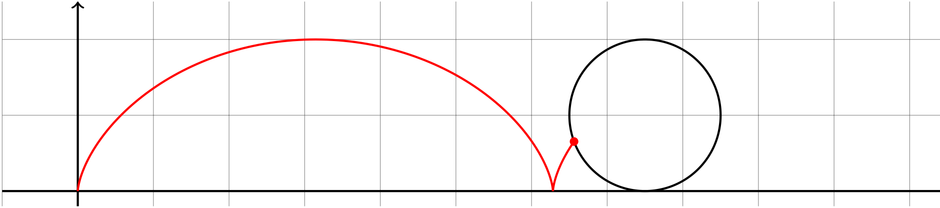
More cusps



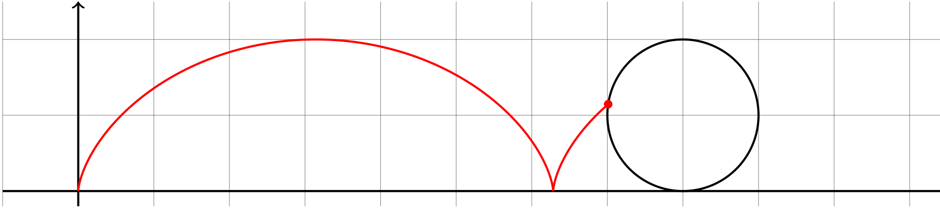
More cusps



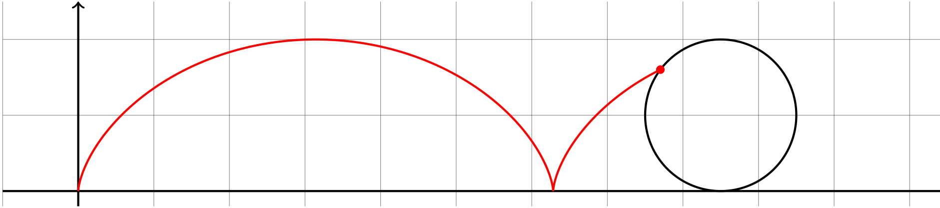
More cusps



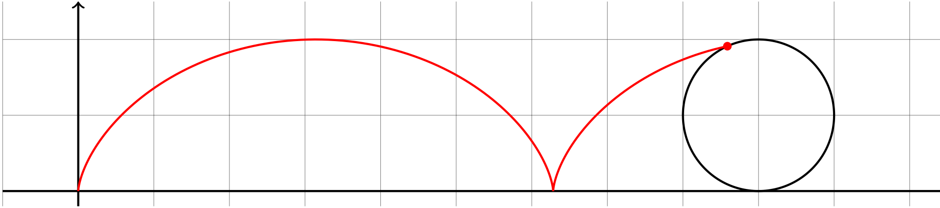
More cusps



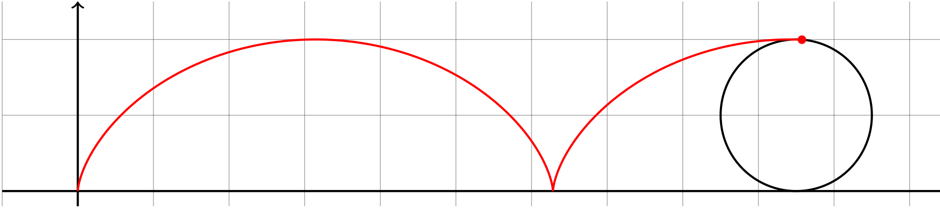
More cusps



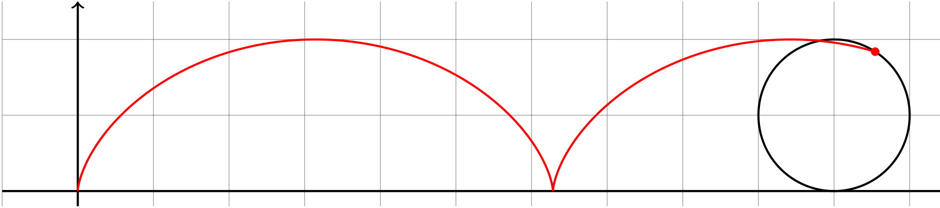
More cusps



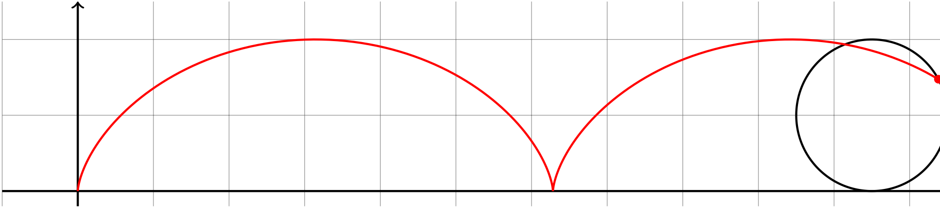
More cusps



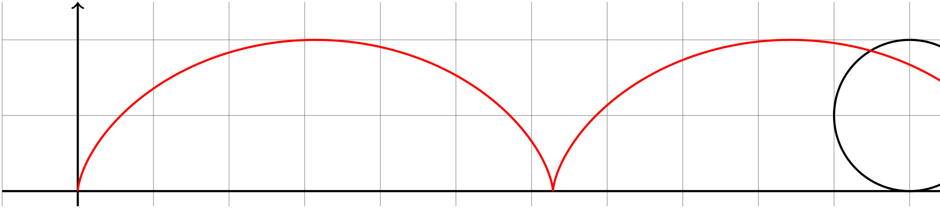
More cusps



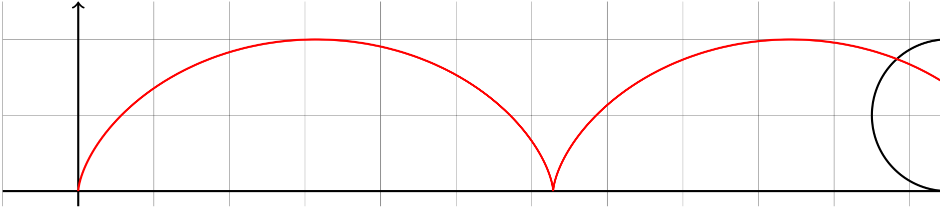
More cusps



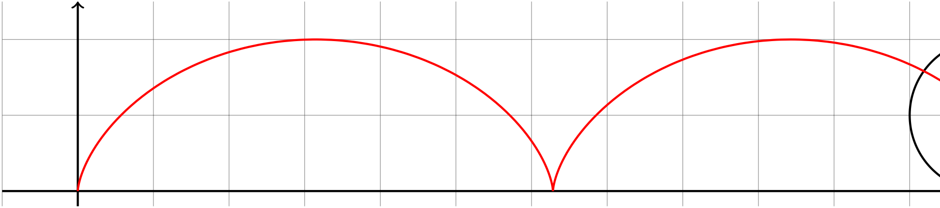
More cusps



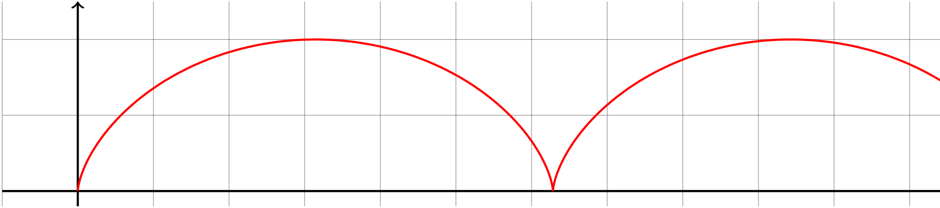
More cusps

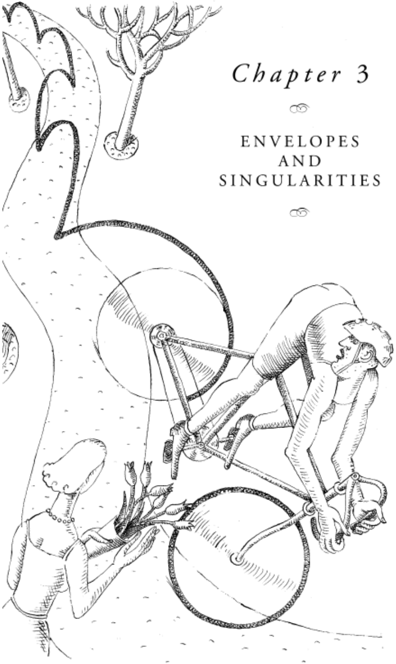


More cusps



More cusps




Chapter 3
ENVELOPES
AND
SINGULARITIES

The curve that was just traced out is called the *cycloid*.

The curve that was just traced out is called the *cycloid*. It is parametrised by

$$x = r(t - \sin t)$$
$$y = r(1 - \cos t)$$

The curve that was just traced out is called the *cycloid*. It is parametrised by

$$x = r(t - \sin t)$$
$$y = r(1 - \cos t)$$

Proposition

The cycloid is not an algebraic curve.

[Video: a bike with “triangular” wheels]

Video: a bike with “triangular” wheels

Video: drilling a square hole

Video: a bike with “triangular” wheels

Video: drilling a square hole

Fun fact: the solution set of the following degree-8 polynomial also has constant width:

$$(x^2+y^2)^4 - 45(x^2+y^2)^3 - 41283(x^2+y^2)^2 + 7950960(x^2+y^2) + 16(x^2-3y^2)^3 + 48(x^2+y^2)(x^2-3y^2)^2 + x(x^2-3y^2)[16(x^2+y^2)^2 - 5544(x^2+y^2) + 266382] - 720^3$$

Rabinowitz, Stanley (1997). A Polynomial Curve of Constant Width. *Missouri Journal of Mathematical Sciences*. 9, pp. 23–27.

Video: a bike with “triangular” wheels

Video: drilling a square hole

Fun fact: the solution set of the following degree-8 polynomial also has constant width:

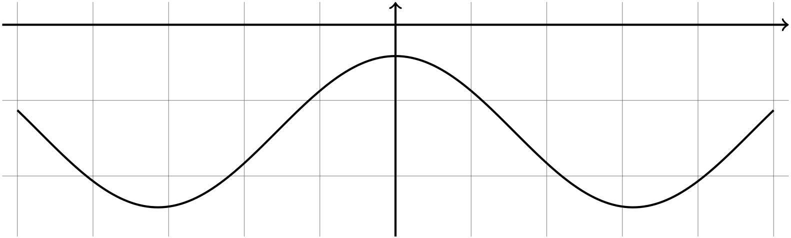
$$
\begin{aligned}
&(x^2+y^2)^4 - 45(x^2+y^2)^3 - 41283(x^2+y^2)^2 + 7950960(x^2+y^2) \\
&\quad + 16(x^2-3y^2)^3 + 48(x^2+y^2)(x^2-3y^2)^2 \\
&\quad + x(x^2-3y^2)[16(x^2+y^2)^2 - 5544(x^2+y^2) + 266382] - 720^3
\end{aligned}
$$

Rabinowitz, Stanley (1997). A Polynomial Curve of Constant Width. *Missouri Journal of Mathematical Sciences*. 9, pp. 23–27.

Video: a bike with square wheels

Wheels

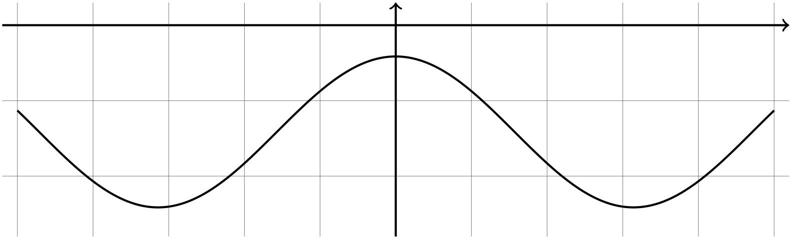
Consider an arbitrary parametric equation for a road, $f(t) = (x(t), y(t))$.



Hall, Leon; Wagon, Stan (1992). Roads and Wheels. *Mathematics Magazine*. 65(5), pp 283–301.

Wheels

Consider an arbitrary parametric equation for a road, $f(t) = (x(t), y(t))$.

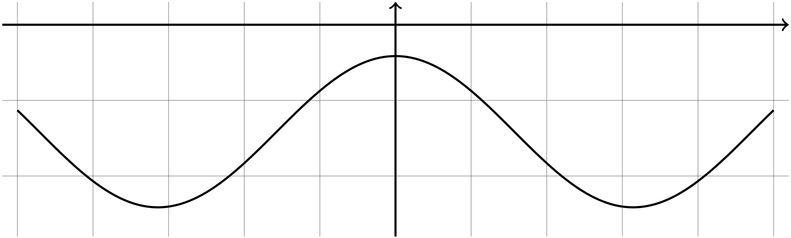


Note that $y(t) < 0$, $x(0) = 0$, and $x(t)$ will be increasing.

Hall, Leon; Wagon, Stan (1992). Roads and Wheels. *Mathematics Magazine*. 65(5), pp 283–301.

Wheels

Consider an arbitrary parametric equation for a road, $f(t) = (x(t), y(t))$.



Note that $y(t) < 0$, $x(0) = 0$, and $x(t)$ will be increasing.
We want to find a wheel that rolls smoothly on the road and whose axis traces the x-axis.

Hall, Leon; Wagon, Stan (1992). Roads and Wheels. *Mathematics Magazine*. 65(5), pp 283–301.

The wheel will be described by a polar function $r(\theta)$, and $\theta = \theta(t)$ describes the rotation of the wheel at time t.

The wheel will be described by a polar function $r(\theta)$, and $\theta = \theta(t)$ describes the rotation of the wheel at time t.
The wheel will be found by solving for

- The initial state: $\theta(0) = -\frac{\pi}{2}$.

The wheel will be described by a polar function $r(\theta)$, and $\theta = \theta(t)$ describes the rotation of the wheel at time t.
The wheel will be found by solving for

- The initial state: $\theta(0) = -\frac{\pi}{2}$.
- The radius condition: $r(\theta(t)) = -y(t)$.

The wheel will be described by a polar function $r(\theta)$, and $\theta = \theta(t)$ describes the rotation of the wheel at time t.
The wheel will be found by solving for

- The initial state: $\theta(0) = -\frac{\pi}{2}$.
- The radius condition: $r(\theta(t)) = -y(t)$.
- The rolling condition:

$$\int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_{-\frac{\pi}{2}}^{\theta(t)} \sqrt{r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

First differentiating the radius condition, $r(\theta(t)) = -y(t)$, gives

$$\frac{dr}{d\theta}\frac{d\theta}{dt} = -\frac{dy}{dt}.$$

First differentiating the radius condition, $r(\theta(t)) = -y(t)$, gives

$$\frac{dr}{d\theta}\frac{d\theta}{dt} = -\frac{dy}{dt}.$$

Now taking the rolling condition,

$$\int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_{-\frac{\pi}{2}}^{\theta(t)} \sqrt{r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta,$$

First differentiating the radius condition, $r(\theta(t)) = -y(t)$, gives

$$\frac{dr}{d\theta}\frac{d\theta}{dt} = -\frac{dy}{dt}.$$

Now taking the rolling condition,

$$\int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_{-\frac{\pi}{2}}^{\theta(t)} \sqrt{r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta,$$

differentiating both sides with respect to t

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \left(\frac{d\theta}{dt}\right) \sqrt{r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2}$$

First differentiating the radius condition, $r(\theta(t)) = -y(t)$, gives

$$\frac{dr}{d\theta}\frac{d\theta}{dt} = -\frac{dy}{dt}.$$

Now taking the rolling condition,

$$\int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_{-\frac{\pi}{2}}^{\theta(t)} \sqrt{r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta,$$

differentiating both sides with respect to t and then squaring gives

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{d\theta}{dt}\right)^2 \left(r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2\right)$$

First differentiating the radius condition, $r(\theta(t)) = -y(t)$, gives

$$\frac{dr}{d\theta}\frac{d\theta}{dt} = -\frac{dy}{dt}.$$

Now taking the rolling condition,

$$\int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_{-\frac{\pi}{2}}^{\theta(t)} \sqrt{r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta,$$

differentiating both sides with respect to t and then squaring gives

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= \left(\frac{d\theta}{dt}\right)^2 \left(r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2\right) \\ &= \left(\frac{d\theta}{dt}\right)^2 r(\theta)^2 + \left(\frac{dy}{dt}\right)^2\end{aligned}$$

So, we have

$$\left(\frac{dx}{dt}\right)^2 = \left(\frac{d\theta}{dt}\right)^2 r(\theta)^2,$$

So, we have

$$\left(\frac{dx}{dt}\right)^2 = \left(\frac{d\theta}{dt}\right)^2 r(\theta)^2,$$

and since everything is positive,

$$\frac{d\theta}{dt} = \frac{dx}{dt} \cdot \frac{1}{r(\theta)} = -\frac{dx}{dt} \cdot \frac{1}{y(t)}$$

So, we have

$$\left(\frac{dx}{dt}\right)^2 = \left(\frac{d\theta}{dt}\right)^2 r(\theta)^2,$$

and since everything is positive,

$$\frac{d\theta}{dt} = \frac{dx}{dt} \cdot \frac{1}{r(\theta)} = -\frac{dx}{dt} \cdot \frac{1}{y(t)}$$

If $x(t) = t$, then

$$\frac{d\theta}{dt} = -\frac{1}{y(t)}$$

So, we have

$$\left(\frac{dx}{dt}\right)^2 = \left(\frac{d\theta}{dt}\right)^2 r(\theta)^2,$$

and since everything is positive,

$$\frac{d\theta}{dt} = \frac{dx}{dt} \cdot \frac{1}{r(\theta)} = -\frac{dx}{dt} \cdot \frac{1}{y(t)}$$

If $x(t) = t$, then

$$\frac{d\theta}{dt} = -\frac{1}{y(t)}$$

If $\theta(t)$ is invertible to $t(\theta)$, then the wheel is given by

$$r(\theta) = -y(t(\theta)).$$

This can also solve the problem: given a wheel, what is the road?

This can also solve the problem: given a wheel, what is the road? Assume that $x(t) = t$, and then consider $r(\theta) = -\csc\theta$; a straight line.

This can also solve the problem: given a wheel, what is the road? Assume that $x(t) = t$, and then consider $r(\theta) = -\csc\theta$; a straight line. Solving

$$\frac{d\theta}{dx} = \frac{1}{r(\theta)} = -\sin\theta$$

This can also solve the problem: given a wheel, what is the road? Assume that $x(t) = t$, and then consider $r(\theta) = -\csc\theta$; a straight line. Solving

$$\frac{d\theta}{dx} = \frac{1}{r(\theta)} = -\sin\theta$$

eventually gives

$$\theta(x) = -2\arctan e^{-x},$$

This can also solve the problem: given a wheel, what is the road? Assume that $x(t) = t$, and then consider $r(\theta) = -\csc\theta$; a straight line. Solving

$$\frac{d\theta}{dx} = \frac{1}{r(\theta)} = -\sin\theta$$

eventually gives

$$\theta(x) = -2\arctan e^{-x},$$

and then plugging it into $y(x) = -r(\theta(x))$ gives

$$y(x) = \csc(-2\arctan e^{-x}) = [...] = -\cosh x$$