Regular double p-algebras (appendix 1)

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La Trobe University

GA Seminar
April 9 2018

A graph:

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A subgraph:

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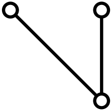

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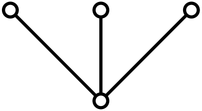

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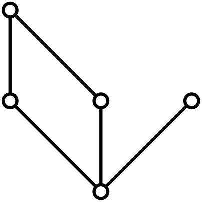

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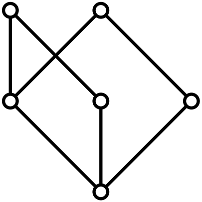

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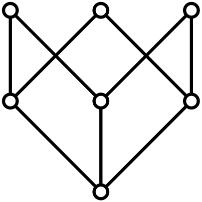

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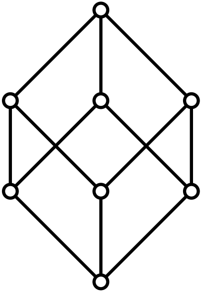

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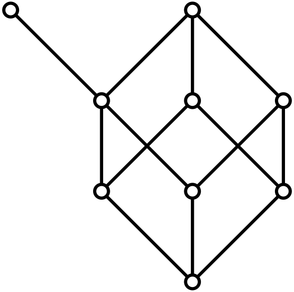

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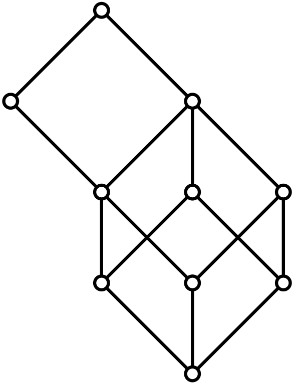

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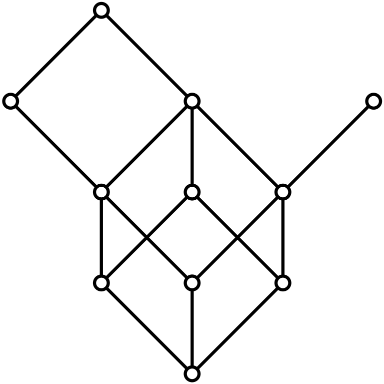

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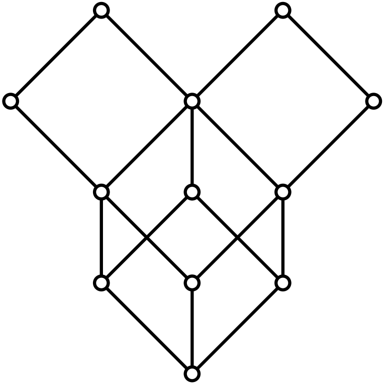

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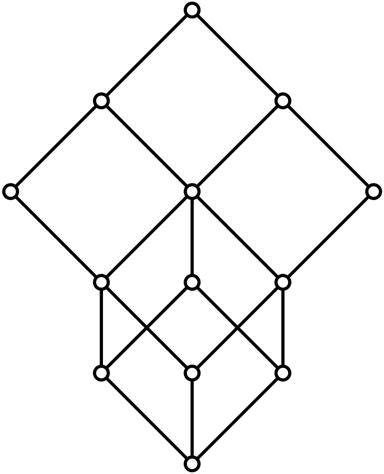

Subgraph lattices

Let $G = \langle V, E \rangle$ be a graph. The set of all subgraphs of G, ordered by inclusion, is a bounded distributive lattice, where

$$\langle V_1, E_1 \rangle \vee \langle V_2, E_2 \rangle = \langle V_1 \cup V_2, E_1 \cup E_2 \rangle$$
$$\langle V_1, E_1 \rangle \wedge \langle V_2, E_2 \rangle = \langle V_1 \cap V_2, E_1 \cap E_2 \rangle.$$

The bounds are given by $0 = \langle \varnothing, \varnothing \rangle$ and $1 = G$. The lattice will be denoted by $\mathcal{S}(G)$.

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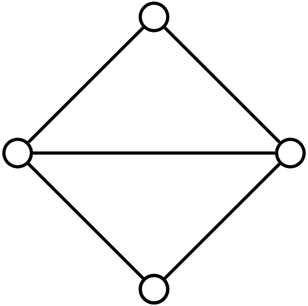
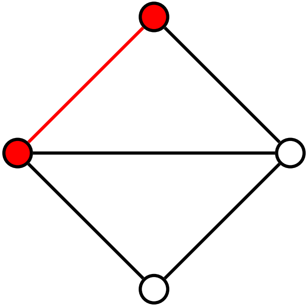
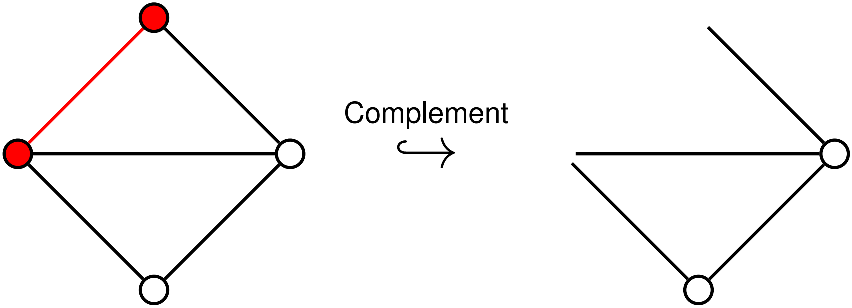
Proposition (Reyes & Zolfaghari, 1996)

Let G be a graph. The lattice $\mathcal{S}(G)$ forms a double Heyting algebra.



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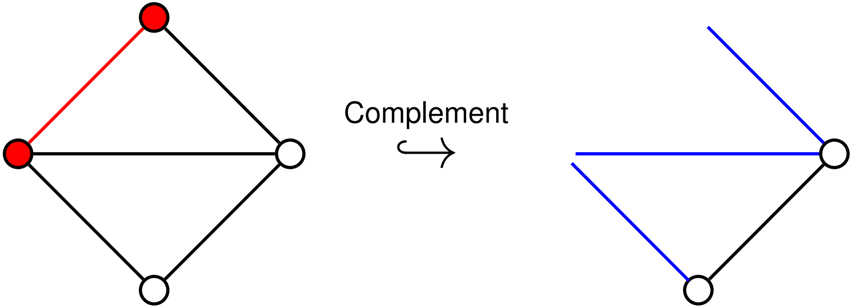
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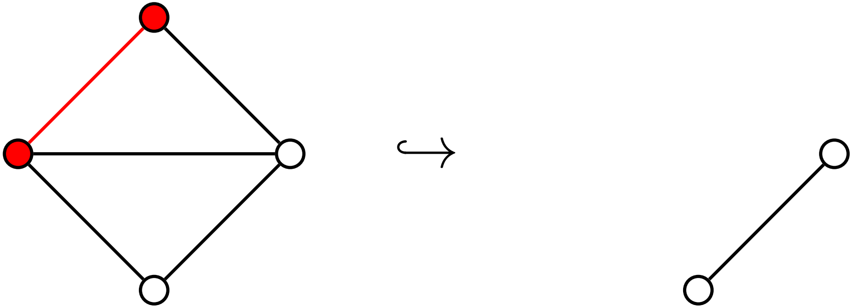

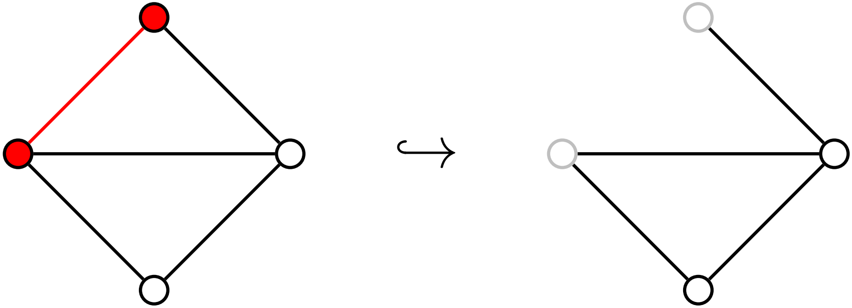

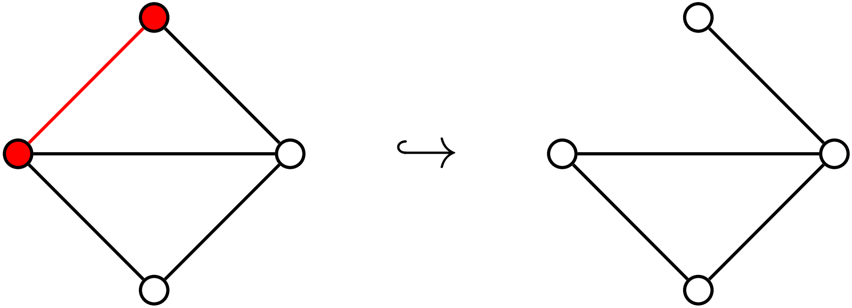

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Similarly, a unary operation $\sim$ is a *dual pseudocomplement* if it satisfies

$$\sim x = \min\{z \in L \mid x \vee z = 1\}.$$

Double p-algebras

Definition

A (distributive) double p-algebra is an algebra $\mathbf{A} = \langle A, \vee, \wedge, \neg, \sim, 0, 1\rangle$ such that

1. $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded (distributive) lattice,
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Theorem

Let G be a graph. Then $\langle \mathcal{S}(G), \cup, \cap, \varnothing, G \rangle$ is the underlying lattice of a distributive double p-algebra.

Pseudocomplements of subgraphs

Take the set complement of the subgraph and abandon the extra edges. Formally, for a graph $G = \langle V, E \rangle$ and a subgraph $H = \langle V', E' \rangle$,

$$\neg H = \langle V \backslash V', \{e \in E \backslash E' \mid (\forall x \in e)\ x \in V \backslash V'\} \rangle.$$

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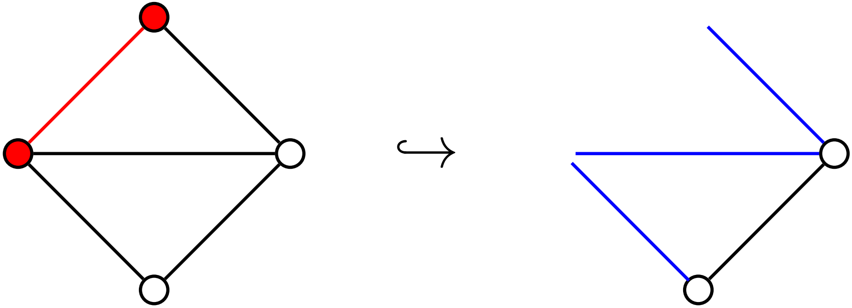

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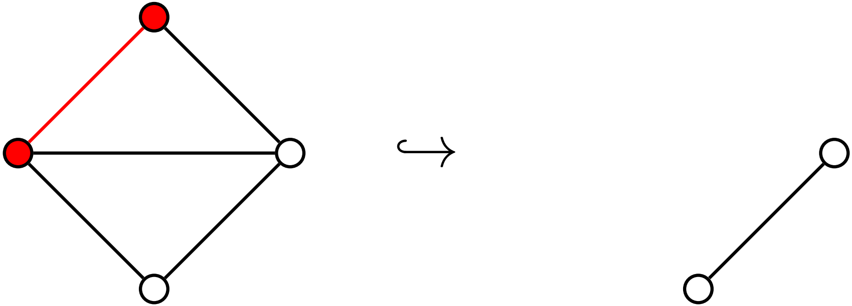

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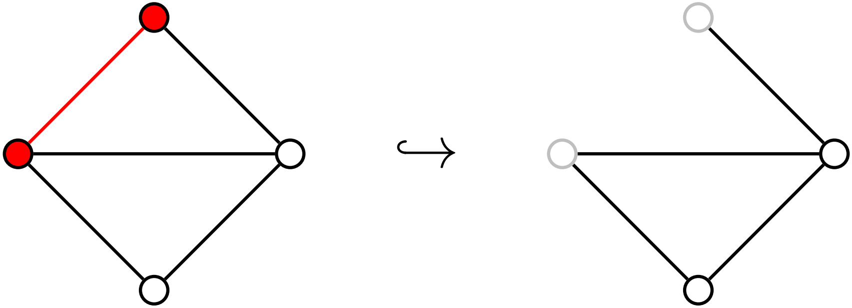

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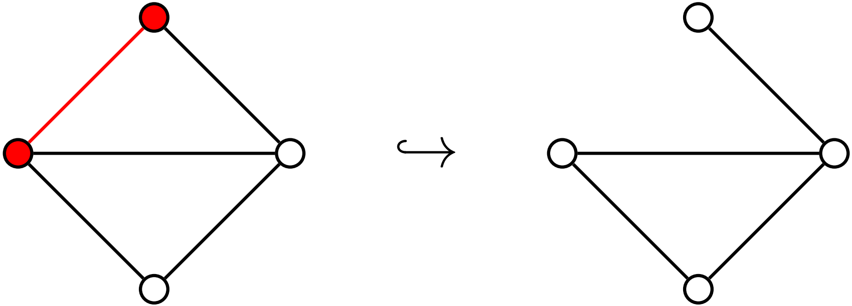

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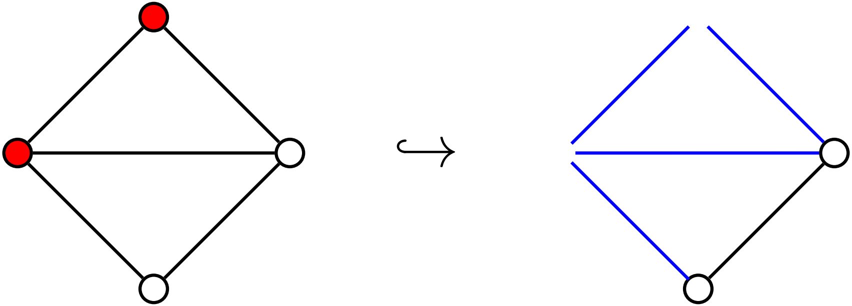

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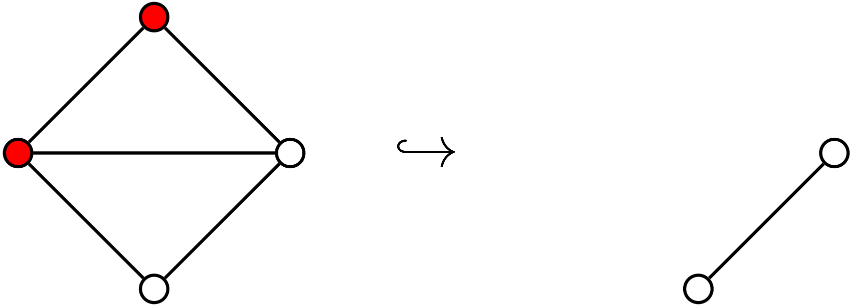

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This definitely does not hold for double p-algebras in general.

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Theorem (Varlet, 1972; Katriňák, 1973)

Let **A** *be a double p-algebra. The following are equivalent*:

1. **A** *is congruence-regular*;
2. *for all* $x, y \in A$, *if* $\neg x = \neg y$ *and* $\sim x = \sim y$, *then* $x = y$;
3. *every prime filter of* **A** *is minimal or maximal*;
4. **A** *is distributive and* $\mathbf{A} \models x \wedge \sim x \leq y \vee \neg y$.

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Theorem (Katriňák, 1973)

Let **A** *be a double p-algebra. If* **A** *is regular, then* **A** *is term-equivalent to a double Heyting algebra via the term*

$$x \rightarrow y = \neg\neg(\neg x \vee \neg\neg y) \wedge [{\sim}(x \vee \neg x) \vee \neg x \vee y \vee \neg y]$$

and its dual.

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Proposition

In every distributive p-algebra, the following identity holds:

$$\neg\neg(\neg x \vee \neg\neg y) \wedge (z \vee \neg x \vee \neg y) \approx \neg\neg(\neg x \vee y) \wedge (z \vee \neg x).$$

Splittings

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A pair of elements (a, b) from a lattice **L** is a *splitting pair* if $a \not\leq b$ and $\uparrow a \cup \downarrow b = L$.

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A pair of elements (a, b) from a lattice $\mathbf{L}$ is a *splitting pair* if $a \not\leq b$ and $\uparrow a \cup \downarrow b = L$.

Definition

A subdirectly irreducible algebra $\mathbf{A}$ in a variety $\mathcal{V}$ is a *splitting algebra* if there exists a subvariety $\mathcal{B}$ of $\mathcal{V}$ such that $(\mathrm{Var}(\mathbf{A}), \mathcal{B})$ is a splitting pair in the lattice of subvarieties of $\mathcal{V}$.

From my thesis:

Theorem

The only finite splitting algebras in the variety of double Heyting algebras are **2** *and* **3**.

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Lemma (McKenzie, 1972)

If a variety $\mathcal{V}$ is congruence-distributive and generated by its finite members, then every splitting algebra in $\mathcal{V}$ is finite.

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Consider an identity $s \approx t$ that fails in a double Heyting algebra **A.** Denote the set of subterms of s or t by Σ. Since $s \approx t$ fails in **A**, there is a tuple $\overline{a}$ of elements from **A** such that $s^{\mathbf{A}}(\overline{a}) \neq t^{\mathbf{A}}(\overline{a})$.

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Finite embeddability property

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A class $\mathcal{K}$ of algebras has the *finite embeddability property* if, for every algebra **A** in $\mathcal{K}$ and every finite partial subalgebra **B** of **A**, there is a *finite* algebra **C** in $\mathcal{K}$ such that **B** embeds as a partial algebra into **C.**

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Proposition

A variety with the finite embeddability property is generated by its finite members.

Partial algebras to sets of terms

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$$B \subseteq \Sigma'(\overline{x}) := \{\sigma(\overline{x}) \mid \sigma \in \Sigma'\}$$

Now close Σ' under subterms and call the result Σ. It is still finite, and we have

$$B \subseteq \Sigma'(\overline{x}) \subseteq \Sigma(\overline{x}).$$

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Our aim: show that $\langle Q; \leq^Q \rangle$ is a finite ordered set such that $\mathcal{U}(Q)$ underlies a regular double p-algebra, and that $\Sigma(\overline{a})$ embeds into $\mathcal{U}(Q)$.

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For all $x, y \in P$, *if* $[x] \leq^Q [y]$, *then* $T(x) \subseteq T(y)$.

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Proposition

The structure $\langle Q; \leq^Q \rangle$ is a finite ordered set and every element of Q is minimal or maximal.

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Proposition

The structure $\langle Q; \leq^Q \rangle$ *is a finite ordered set and every element of* Q *is minimal or maximal.*

Theorem

The partial algebra $\Sigma(\overline{a})$ *embeds as a partial algebra into* $\mathcal{U}(Q)$.

Corollary

The variety of regular double p-algebras has the finite embeddability property; hence it is generated by its finite members.

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The only splitting algebras in the variety of regular double p-algebras are **2** *and* **3**.