Semisimple varieties

Christopher J. Taylor

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Previously

Let **A** be an algebra with a Heyting algebra reduct.

- To every congruence θ on **A** there must be a filter F such that

  $$\theta = \theta(F) := \{(x, y) \mid x \leftrightarrow y \in F\},$$

  where $x \leftrightarrow y = x \to y \wedge y \to x$.
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- If **A** possesses a unary term t such that, for every filter F,

  $$\theta(F) \text{ is a congruence if and only if } F \text{ is closed under } t,$$

  then that term is called a congruence-filter term on **A**.

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For convenience reasons, we also demand that a congruence-filter term is order-preserving. This will be assumed henceforth.

We also define the pseudocomplement operation by $\neg x = x \to 0$.

In my last talk, I gave an overview of a general method that constructs congruence-filter terms under certain natural conditions. It is based on a technique which, for any finitary operation f on a Heyting algebra, produces a unary partial operation $[f]$.

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Proposition (Hasimoto, 2001)

Let $\mathbf{A}$ be a Heyting algebra and let f be an operation on A. If $[f]$ is a total operation, then $[f]$ is meet-preserving and 1-absorbing.

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Proposition (Hasimoto, 2001)

Let $\mathbf{A}$ be a Heyting algebra and let f be an operation on A. If $[f]$ is a total operation, then $[f]$ is meet-preserving and 1-absorbing.

Theorem (T., 2017)

Let $\mathbf{A} = \langle A; \vee, \wedge, \rightarrow, f_1, f_2, \ldots, f_n, 0, 1 \rangle$ be an expanded Heyting algebra. If each $[f_i]$ is a term function in the language of $\mathbf{A}$, then

$$dx := x \wedge \bigwedge \{[f_i]x \mid i \leqslant n\}$$

is a congruence-filter term on $\mathbf{A}$.

For simplicity, here we restrict attention to unary operations.

- Suppose f is a unary operation on a Heyting algebra.
  - If f is meet-preserving and 1-absorbing, then $[f]x = fx$.
  - If f is meet-reversing and 1-reversing, then $[f]x = \neg fx$.

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- H^+-algebras (Sankappanavar, 1985)
  - $\mathbf{A} = \langle A; \vee, \wedge, \rightarrow, \sim, 0, 1 \rangle$, where $\sim$ is a dual pseudocomplement. As $\sim$ is meet-reversing and 1-reversing, $\neg{\sim}x$ is a congruence-filter term.

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- Ockham–Heyting algebras (Sankappanavar, 1987)
  - $\mathbf{A} = \langle A; \vee, \wedge, \rightarrow, \frown, 0, 1\rangle$, where $\frown$ is a dual lattice endomorphism. Then $\frown$ is meet- and 1-reversing by definition, so $\neg{\frown}x$ is a congruence-filter term.

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- De Morgan–Heyting algebras (Montiero, 1980)
  - An Ockham–Heyting algebra such that ${\frown}{\frown}x = x$; in other words, $\frown$ is an involutive dual lattice automorphism.

Some cases do not follow from general results, but can be proved directly.

- Double Heyting algebras (Köhler, 1980)
  - $\mathbf{A} = \langle A; \vee, \wedge, \rightarrow, \dot{-}, 0, 1\rangle$. One can show directly that $[\dot{-}]x = \neg{\sim}x$.

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- Monadic Heyting algebras (Bezhanishvili, 1995)
  - $\mathbf{A} = \langle A; \vee, \wedge, \rightarrow, \forall, \exists, 0, 1 \rangle$. $\forall$ is meet-preserving and 1-absorbing, and $\exists$ is join-preserving and 0-absorbing. Join-preserving operations are not covered by the previous result, but one can show, using the fact that $\forall(x \rightarrow y) \leqslant \exists x \rightarrow \exists y$, that the $\langle \vee, \wedge, \rightarrow, \forall, 0, 1 \rangle$-reduct is sufficient for congruences, and then $dx = \forall x$ is a congruence-filter term.

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Some subvarieties have also been considered; e.g., Stone double Heyting algebras (Iturrioz, 1976) and De Morgan–Heyting algebras satisfying $\neg\neg x = \frown\neg x$ (Meskhi, 1982).

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Each of these authors characterised congruences by means of a unary term, and several considered a treatment of semisimple varieties and discriminator varieties.

A unary operation t is descending if $tx \leqslant x$, for all x.

Any congruence-filter term t can be made descending: let $dx = x \wedge tx$. Then d is a congruence-filter term if and only if t is a congruence-filter term.

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Theorem (Congruence theorem)

Let $\mathbf{A}$ be an algebra with a descending congruence-filter term d.

- *$\mathbf{A}$ has the congruence extension property.*
- *$\mathbf{A}$ is subdirectly irreducible if and only if there exists $b \in A \backslash \{1\}$ such that, for all $x \in A \backslash \{1\}$, there is some $n \in \omega$ such that $b \geqslant d^n x$.*
- *$\mathbf{A}$ is simple if and only if, for all $x \in A \backslash \{1\}$, there exists $n \in \omega$ such that $d^n x = 0$.*

For a variety $\mathcal{V}$ and any (possibly empty) adjective `BLANK`, a unary term t is a `BLANK` congruence-filter term on $\mathcal{V}$ if it is a `BLANK` congruence-filter term on every algebra in $\mathcal{V}$.

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Theorem (T., 2017)

Let $\mathcal{V}$ be a variety of algebras with a descending congruence-filter term d. The following are equivalent:

1. *$\mathcal{V}$ has definable principal congruences (DPC);*
2. *$\mathcal{V}$ has equationally definable principal congruences (EDPC);*
3. *there exists $n \in \omega$ such that $\mathcal{V} \models d^{n+1}x = d^n x$.*

Let $\mathcal{V}$ be a variety in any signature.

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- If there is a term t in the language of $\mathcal{V}$ such that the corresponding term function satisfies

$$t^{\mathbf{A}}(x, y, z) = \begin{cases} x & \text{if } x \neq y, \\ z & \text{if } x = y, \end{cases}$$

  for every subdirectly irreducible algebra $\mathbf{A}$ in the variety, then $\mathcal{V}$ is a discriminator variety.

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Theorem (Blok, Köhler, Pigozzi, 1984)

The following are equivalent:

1. *$\mathcal{V}$ is semisimple, congruence-permutable, and has EDPC;*
2. *$\mathcal{V}$ is a discriminator variety.*

Heyting algebras have a Mal'cev term:

$$p(x, y, z) = [(x \to y) \to z] \wedge [(z \to y) \to x].$$

Corollary

If $\mathcal{V}$ has a Heyting algebra reduct, then the following are equivalent:

1. *$\mathcal{V}$ is semisimple and has EDPC;*
2. *$\mathcal{V}$ is a discriminator variety.*

Definition

An H$^+$-algebra is an algebra $\mathbf{A} = \langle A; \vee, \wedge, \to, \sim, 0, 1\rangle$ such that

- $\langle A; \vee, \wedge, \to, 0, 1\rangle$ is a Heyting algebra, and
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As $\neg{\sim}x \leqslant x$ is always true, a strongly descending operation is descending.

Furthermore, if an algebra with an H$^+$-algebra reduct has a congruence-filter term t, then it has a strongly descending one:

$$dx = \neg{\sim}x \wedge tx.$$

Theorem (T., 2017)

Let $\mathcal{V}$ be a variety with an H^+-algebra reduct and a strongly descending congruence-filter term d. If $\mathcal{V}$ is semisimple, then there exists $n \in \omega$ such that $\mathcal{V} \models d^{n+1}x = d^n x$.

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Of the algebras listed earlier, only monadic Heyting algebras and Ockham–Heyting algebras do not meet the hypothesis of this theorem. Note that the De Morgan–Heyting subvarieties of OH are fine, though.

Theorem (T., 2017)

Let $\mathcal{V}$ be a variety of algebras with an H^+-algebra term-reduct and a strongly descending congruence-filter term d. The following are equivalent:

1. $\mathcal{V}$ *is semisimple*;
2. $\mathcal{V}$ *is a discriminator variety*;
3. $\mathcal{V}$ *has DPC and* $\mathcal{V} \models x \leqslant d{\sim}d^n\neg x$, *for some* $n \in \omega$;
4. $\mathcal{V}$ *has EDPC and* $\mathcal{V} \models x \leqslant d{\sim}d^n\neg x$, *for some* $n \in \omega$;
5. $\mathcal{V} \models d^{n+1}x = d^n x$ *and* $\mathcal{V} \models x \leqslant d{\sim}d^n\neg x$, *for some* $n \in \omega$;
6. $\mathcal{V} \models d^{n+1}x = d^n x$ *and* $\mathcal{V} \models d{\sim}d^n x = {\sim}d^n x$, *for some* $n \in \omega$.

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Theorem (T., 2016)

Let $\mathcal{V}$ be a variety of double Heyting algebras. The following are equivalent:

1. *$\mathcal{V}$ is semisimple;*
2. *$\mathcal{V}$ is a discriminator variety;*
3. *$\mathcal{V}$ has DPC;*
4. *$\mathcal{V}$ has EDPC;*
5. $\mathcal{V} \models d^{n+1}x = d^n x$.

We can define the dual pseudocomplement on De Morgan–Heyting algebras by $\sim x = \frown\neg\frown x$. We then have a strongly descending congruence-filter term $dx = \neg\sim x \wedge \neg\frown x$.

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It is then natural to ask, are either of these two varieties semisimple?
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Proposition

Let **A** *be an algebra with a strongly descending congruence-filter term d.*

- *For all $x \in A$, if $dx = x$, then x is complemented.*
- *If there exists $x \in A \backslash \{0, 1\}$ such that $dx = x$ and $d{\sim}x = {\sim}x$, then* **A** *is not subdirectly irreducible.*

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Proof.

If $x = dx$, then $x = dx \leqslant \neg{\sim}x \leqslant x$, so $x = \neg{\sim}x$.

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If $dx = x$, the congruence-filter generated by x is the filter generated by x, and similarly for ${\sim}x$. Then ${\uparrow}x \cap {\uparrow}{\sim}x = \{1\}$, which supplies two congruences with a trivial intersection. □

Proposition

Let $\mathbf{A}$ *be an algebra with a strongly descending congruence-filter term* d*. Assume* $\mathbf{A} \models dx = x \Rightarrow d{\sim}x = {\sim}x$ *and* $\mathbf{A}$ *satisfies the decending chain condition. If* $\mathbf{A}$ *is subdirectly irreducible, then* $\mathbf{A}$ *is simple.*

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Consider the descending chain

$$b \geqslant db \geqslant d^2 b \geqslant d^3 b \geqslant \ldots \geqslant d^n b \geqslant \ldots$$

By DCC, there exists $n \in \omega$ such that $d^{n+1} b = d^n b$.

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A subdirectly irreducible double Heyting algebra which is not simple

PETER KÖHLER

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A SUBDIRECTLY IRREDUCIBLE SYMMETRIC HEYTING ALGEBRA WHICH IS NOT SIMPLE

A. GALLI and M. SAGASTUME

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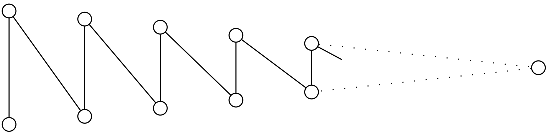

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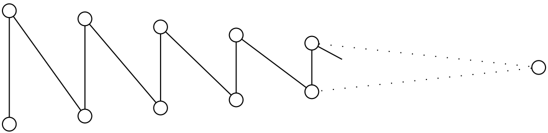

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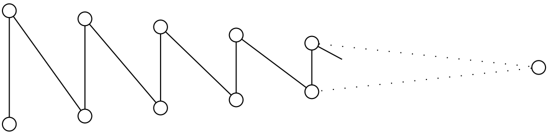

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It is sufficient to exhibit a sequence of algebras $\langle \mathbf{A}_n \rangle_{n\in\omega}$ such that $\mathbf{A}_n \not\models d^{n+1}x = d^n x$. We can even construct such a sequence consisting only of simple algebras!

Definition

A symmetric Heyting relation algebra (SHRA) is an algebra $\langle A; \vee, \wedge, \to, \circ, \frown, 0, 1, e\rangle$ such that:

1. $\langle A; \vee, \wedge, \to, \frown, 0, 1\rangle$ is a De Morgan–Heyting algebra,
2. $\langle A; \circ, e\rangle$ is a monoid,
3. (a) $(\forall x, y, z \in A)\ x \circ y \leqslant z \iff x \leqslant \frown(y \circ \frown z)$.
   (b) $(\forall x, y \in A)\ {\sim}\frown(x \circ y) \leqslant ({\sim}\frown y) \circ ({\sim}\frown x)$,

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We then obtain a congruence-filter term for SHRAs:

$$dx = \neg{\sim}x \wedge \neg{\frown}x \wedge \neg(1 \circ {\sim}x) \wedge \neg({\sim}x \circ 1).$$

SHRAs were introduced by Stell (2014) to generalise relation algebras from sets to graphs.

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- If $\mathbf{A}$ is a Boolean SHRA, then we can define a relation algebra by $\neg x = x \to 0$ and ${\smile}x = \neg {\frown} x$.

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In both cases, the lattice and monoid structure remains unchanged.

Relation algebras are a discriminator variety. The typical proof of this relies on the fact that subdirectly irreducible relation algebras satisfy

$$1 \circ x \circ 1 = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \neq 0. \end{cases}$$

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In pursuit of this, a reliable means of constructing SHRAs would be useful.

An error in my thesis!?!

Open Problem 11. Is $\mathcal{SHRA}$ a discriminator variety? One way to prove otherwise is to observe that, because of the double Heyting algebra reduct, finite subdirectly irreducible SHRAs are simple. Then, to show that $\mathcal{SHRA}$ is not a discriminator variety, it would be sufficient to exhibit, for each $n \in \omega$, a finite SHRA—perhaps based on a fence—that does not satisfy $d^n x = 0$.

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Open problem

Does every De Morgan–Heyting algebra underlie an SHRA?

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The binary relations on a set S are in one-to-one correspondence with the set of functions on $\wp(S)$ which preserve arbitrary joins.

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Thus, Stell considers $\bigvee$-preserving functions on lattices of *subgraphs* instead of *subsets*.

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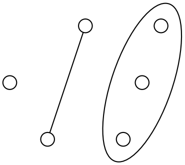
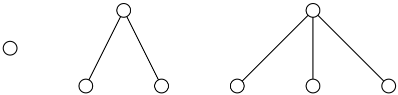
- The nonminimal elements are the edges.
- The nonmaximal elements are the vertices.
- Elements which are both maximal and minimal are treated, without any loss of generality, as either empty edges or isolated vertices.

A vertex v lies on an edge e if and only if $v < e$.





Definition

Let φ be a preorder on U and let x be a binary relation on U.
If $\varphi \circ x \circ \varphi = x$, then we say that x is φ-stable.

Stell showed that, for a hypergraph $\langle U; \varphi \rangle$, the φ-stable relations on U correspond to the $\bigvee$-preserving maps on the lattice of subgraphs of $\langle U; \varphi \rangle$.

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Let φ be a preorder on a set U. Denote the set of φ-stable relations by

$$\mathcal{R}_\varphi(U) = \{x \in \mathcal{R}(U) \mid \varphi \circ x \circ \varphi = x\},$$

where $\mathcal{R}(U)$ is the set of all binary relations on U.

Theorem (Stell, 2014)

Let φ be a preorder on a set U.

- $\{\varnothing, \varphi, U \times U\} \subseteq \mathcal{R}_\varphi(U)$.
- *If* $x \in \mathcal{R}_\varphi(U)$, *then* $x \circ \varphi = \varphi \circ x = x$.
- $\mathcal{R}_\varphi(U)$ *is closed under arbitrary unions and intersections.*
- $\mathcal{R}_\varphi(U)$ *is closed under relational composition.*
- $\mathcal{R}_\varphi(U)$ *is closed under the converse-complement operation,* $\frown x := \smile(x') = (\smile x)'$.
- $\mathcal{R}_\varphi(U)$ *is closed under* $\to$, *where* $x \to y = (\smile h \circ (x \wedge y') \circ \smile h)'$.
- $\langle \mathcal{R}_\varphi(U); \vee, \wedge, \to, \frown, 0, 1 \rangle$ *is a De Morgan–Heyting algebra.*
- $\langle \mathcal{R}_\varphi(U); \circ, \varphi \rangle$ *is a monoid.*

Let $\mathbf{R}_\varphi(U)$ denote the algebra $\langle \mathcal{R}_\varphi(U); \vee, \wedge, \to, \circ, \frown, 0, 1, \varphi \rangle$.

Proposition

Let φ be a preorder on a set U.

- *If φ is an equivalence relation, then $\mathcal{R}_\varphi(U) \cong \mathcal{R}(U/\varphi)$.*
- *If $\mathcal{R}_\varphi(U)$ is Boolean, then φ is an equivalence relation.*

Corollary

If a relation algebra is not representable as an algebra of binary relations, then it cannot be represented as an algebra of φ-stable relations.

For a preorder φ, define the corresponding equivalence relation $\sim_\varphi$ and partial order $\leqslant_\varphi$ on equivalence classes in the usual way.

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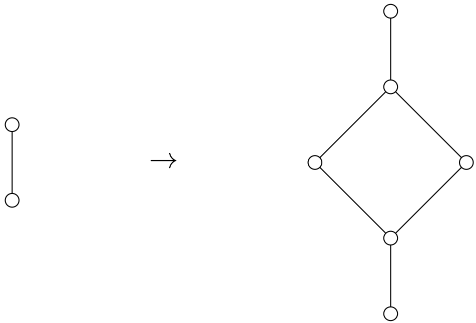

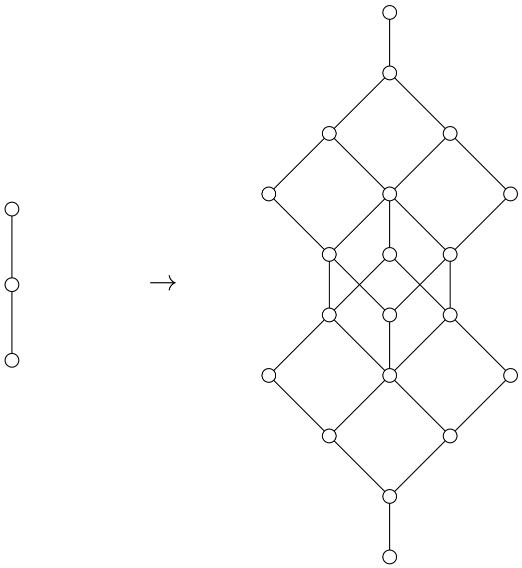

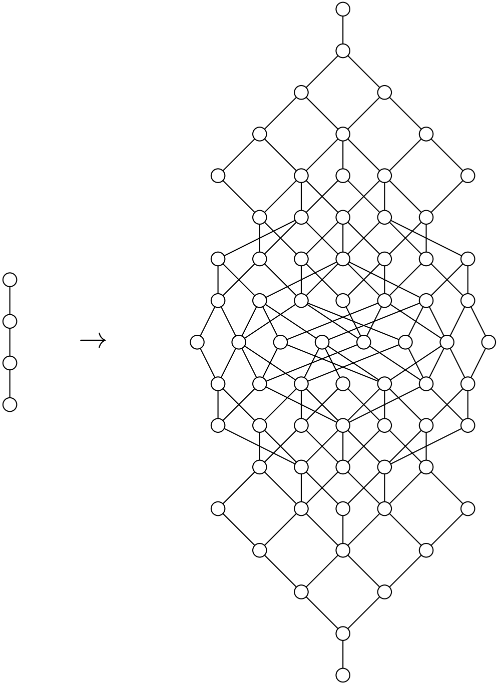

Theorem

Let $\langle U; \leqslant \rangle$ *be an ordered set. Then* $\mathcal{R}_{\leqslant}(U) = \mathcal{O}(U \times U^{\partial})$.